

SPECTRAL ASYMPTOTICS OF SEMICLASSICAL UNITARY OPERATORS

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ABSTRACT. Motivated by an axiomatic semiclassical quantization for self-adjoint operators due to Polterovich, Vũ Ngọc, and the second author, we give an axiomatic quantization for non necessarily self-adjoint operators. Then we prove within this framework that the semiclassical joint spectrum (“quantum spectrum”) of a family of commuting unitary operators satisfying certain assumptions determines the image of its joint principal symbol (“classical spectrum”). This is the first time in which this aspect of Bohr’s correspondence principle is proved for a class of non self-adjoint operators.

1. INTRODUCTION

In the study of the semiclassical limit of quantum mechanics, there are two main contexts depending on the nature of the phase space of the underlying classical systems. If the latter is the cotangent bundle of a Riemannian manifold, the relevant quantization involves the now classical subject of pseudodifferential operators, which has been initiated about fifty years ago, following the works of Hörmander and Duistermaat [16, 12]. The second important context is the one of Berezin-Toeplitz operators, which fit inside the framework of the now well known geometric quantization of Kostant [17] and Souriau [27], which were introduced by Berezin [1], and whose microlocal analysis was initiated by Boutet de Monvel and Guillemin [4]. They appear in the case where the classical phase space is a closed symplectic manifold.

Since these two settings share many common properties, but still need to be dealt with separately when proving these because they use quite different constructions, there has been a will lately to formulate a minimal set of general axioms satisfied by both pseudodifferential and Berezin-Toeplitz operators and sufficient to prove substantial results about them. A first successful attempt was made when the second author, together with Polterovich and Vũ Ngọc [21], introduced such an axiomatic definition of semiclassical self-adjoint operators and used it to prove that the joint spectrum of a family of commuting such operators allows to recover the convex hull of the image of the underlying momentum map in the semiclassical limit. One of the satisfactory aspects of this work is that the axiomatization is minimal, as there

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is no mention to the symplectic form or, of course, of the correspondence principle for commutators and Poisson brackets, and yet it allows to derive this general physical property.

However, it is also interesting to include non self-adjoint operators in the picture. Indeed, such operators appear in partial differential equations, for instance in problems concerning damped wave equations, scattering poles, and convection-diffusion. Furthermore, and this is where our interest in the question mostly came from, semiclassical unitary operators quantize symplectic but non Hamiltonian torus actions on symplectic manifolds. In view of recent work of Tolman [28] where she constructs the first example of a symplectic non Hamiltonian action on a compact connected manifold with isolated fixed points, using a method which shows abundance of such examples, these have become of a great relevance in symplectic geometry. In view of our paper, we expect that semiclassical techniques can shed further light on this area of symplectic geometry. Symplectic non Hamiltonian circle actions do not admit real-valued momentum maps but rather circle-valued momentum maps, and the natural quantization of a circle-valued function is a (quasi-)unitary operator. We should make it clear that we are not talking about Fourier integral operators, which are used to quantize symplectomorphisms, but about genuine pseudodifferential or Berezin-Toeplitz operators quantizing S^1 -valued Hamiltonians.

Motivated by these considerations, we introduce in this paper an axiomatic semiclassical quantization, based on axioms satisfied by both Berezin-Toeplitz and pseudodifferential operators, extending the one of [21] and including non necessarily self-adjoint operators, and then restrict our attention to those operators which are unitary. More precisely, we prove that for commuting unitary operators which satisfy our set of axioms and some generic additional assumptions, the convex hull of the joint spectrum converges to the convex hull of the classical spectrum. This is in accordance with the following manifestation of Bohr's correspondence principle: the quantum mechanics of unitary operators converges to the classical mechanics of their principal symbols in the high frequency limit. The novelty of the paper is that, to our knowledge, it provides with the first rigorous proof of this general principle in the non-selfadjoint case.

Unlike in the self-adjoint case, the joint spectrum of a family of semiclassical unitary operators and the image of its joint momentum map are subsets of the torus $\mathbb{T}^d = (S^1)^d$, and in the following result the notion of convex hull used is a new geometric construction which is subtle enough to, roughly speaking, be applied to very small sets and still give small convex hulls. One might legitimately ask why we even bother considering convex hulls. The problem is that the result is most probably not true if we only consider the quantum and classical spectra instead of their convex hulls; this can be seen from the main result in [21], which deals with self-adjoint operators and also involves convex hulls (and a look at the proofs in this paper convinces that

taking convex hulls is indeed very important). The following is our main theorem (see Theorems 3.1 and 6.6 for detailed versions).

Theorem 1.1 (Correspondence principle for the joint spectrum of unitary operators). *Under generic conditions, the semiclassical limit of the convex hull of the joint spectrum of a collection of pairwise commuting unitary operators converges to the convex hull of the image of the map $F: M \rightarrow \mathbb{T}^d$ whose components are the \mathbb{S}^1 -valued principal symbols of the operators.*

The joint spectrum above is often called the “quantum spectrum”, while the image of the principal symbols is often called the “classical spectrum”. The generic conditions we give are quite weak, nonetheless we are unable to remove them without additional axioms and it is conceivable that the theorem does not hold without these conditions, however, it should hold without introducing too many new axioms and at the end of the paper we formulate a conjecture in this direction. We believe that proving the conjecture poses a more complicated challenge, and overcoming the technical assumptions probably requires additional new ideas and/or techniques; this is why we do not try to do so in this paper, which is already heavy.

We show that the general axioms of semiclassical quantization in this paper (Section 4.2) apply to both pseudodifferential operators and Berezin-Toeplitz operators, which provides with potential applications. To illustrate this, we explicitly compute the example of semiclassical Berezin-Toeplitz quantization of symplectic but non Hamiltonian actions and then give an example in this setting for which our conjecture is satisfied.

The proof of our main result has two interconnected parts; the first part is geometric, and the second part is analytic, and we encounter two main difficulties which spread on both parts. In the first part of the paper we construct the suitable notion of convex hull. The second part of the paper is the analysis of the limit of the quantum joint spectrum. Next we explain the difficulties we encounter.

The first difficulty is that in order to use the result in [21] for self-adjoint operators, which seems to be a fairly natural plan of attack, we want to transform our unitary operators into self-adjoint operators. There is a standard tool to do so, called the Cayley transform, but it can only be applied to unitary operators not containing -1 in their spectrum. Hence, this gives a first restriction on our operators; of course one could argue that this difficulty could be overcome by using a direct route instead of the result on self-adjoint operators, however it is not clear that it would be more efficient. Instead it might be more useful to expand our set of axioms to work with approximations of our operators in order to reach the general case in the future; this part is still unclear.

The second difficulty is that, as we have already explained, both the joint spectrum of a family of semiclassical unitary operators and the image of its joint momentum map are subsets of $\mathbb{T}^d = (\mathbb{S}^1)^d$. Now, in [21], the result is that the convex hull of the joint spectrum of a family of semiclassical

self-adjoint operators (a subset of \mathbb{R}^d) converges to the convex hull of the classical image (another subset of \mathbb{R}^d), and it can be seen from the proofs that it is crucial to consider these convex hulls. Hence the question in our case is: what is the convex hull of a subset of \mathbb{T}^d ? It turns out that this question is rather involved, and the naive idea of defining the convex hull of a subset E of \mathbb{T}^d by lifting E in \mathbb{R}^d , taking the convex hull of the lift and then projecting back is not satisfactory, even if we forget about our spectral problem, because very often, the outcome will be far too large.

Coming back to our problem, this procedure will not give the desired semiclassical limiting theorem because more often than not the resulting subset will be a far too large subset of the torus and the semiclassical spectrum, while inside of it, will be tiny. This is a geometric problem we have to deal with, that does not appear in the self-adjoint case, and takes most of the first part of the paper. It is worth noting that in this setting, we encounter the same problem with the point -1 , as the natural argument has a jump at this point. This similarity with the quantum problem is not really surprising because the Cayley transform acts as taking the argument at the principal symbol level.

Note that the notion of semiclassical quantization that we introduce allows us to handle general non-self-adjoint semiclassical operators. It would be interesting to see if from the family of joint spectra of a family of pairwise commuting normal operators (not necessarily self-adjoint or unitary) one can recover some information about the classical spectrum \mathcal{S} . One can maybe try to recover the convex hull of $\mathcal{S} \subset \mathbb{C}^d \simeq \mathbb{R}^{2d}$, but the analysis seems quite complicated. Furthermore, we would lose information for unitary operators because then the convex hull of $\mathcal{S} \subset \mathbb{C}^d$ would be significantly larger than its convex hull computed as a subset of \mathbb{T}^d .

Outline of the Paper. In Section 2, we recall the notions of classical and quantum spectrum, Hausdorff distance, and the known result on the convergence of the quantum to the classical spectrum for self-adjoint operators. In Section 3, we state our main theorem, proving the convergence of the quantum to the classical spectrum for unitary operators. In Section 4, we construct an axiomatic quantization for non necessarily self-adjoint semiclassical operators, applicable in particular to unitary ones. In Section 5, we construct a new notion of convex hull for subsets of tori which is subtle enough to give “small” convex hull for “small” sets. In Section 6 we prove our main theorem, putting together the ingredients from the previous sections. In Section 7, we apply our main theorem to the unitary operators which quantize non-Hamiltonian symplectic actions.

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2. PRELIMINARES

We start by reviewing the case of self-adjoint operators from [21], but first let us recall the basic terminology used in the study of semiclassical operators. A finite number of normal operators S_1, \dots, S_d on a Hilbert space are said to be *mutually commuting* if their corresponding spectral measures μ_1, \dots, μ_d pairwise commute. In this case we may define the joint spectral measure $\mu := \mu_1 \otimes \dots \otimes \mu_d$ on \mathbb{C}^d .

2.1. Classical spectrum and joint spectrum. In this paper we are concerned with semiclassical operators, that is, the operator itself is given by a sequence of operators, labelled by the Planck constant \hbar , considered as a small parameter. Let I be a subset of $(0, 1]$ that accumulates at 0. Let

$$\mathcal{F} = (T_1 := (T_1(\hbar))_{\hbar \in I}, \dots, T_d := (T_d(\hbar))_{\hbar \in I})$$

be a collection of pairwise commuting semiclassical normal operators. These operators depend on the parameter $\hbar \in I$ and act on a Hilbert space \mathcal{H}_\hbar , $\hbar \in I$. We assume that at each $\hbar \in I$ the operators have a common dense domain $\mathcal{D}_\hbar \subset \mathcal{H}_\hbar$ such that the inclusion $T_j(\hbar)(\mathcal{D}_\hbar) \subset \mathcal{D}_\hbar$ holds for all $j = 1, \dots, d$. For a fixed value of \hbar , the *joint spectrum* of $(T_1(\hbar), \dots, T_d(\hbar))$ is the support of their joint spectral measure. It is denoted by $\text{JointSpec}(T_1(\hbar), \dots, T_d(\hbar))$. For instance, if the Hilbert space \mathcal{H}_\hbar is finite dimensional (which is the case for instance with Berezin-Toeplitz operators on closed symplectic manifolds), then $\text{JointSpec}(T_1(\hbar), \dots, T_d(\hbar))$ is the set of $(\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$ such that there exists $v \neq 0$ satisfying $T_j(\hbar)v = \lambda_j v$ for all $j = 1, \dots, d$.

Definition 2.1. The *joint spectrum* $\text{JointSpec}(T_1, \dots, T_d)$ of (T_1, \dots, T_d) is the collection of all joint spectra of $T_1(\hbar), \dots, T_d(\hbar)$ for $\hbar \in I$.

Now suppose that the connected manifold M has a semiclassical quantization (we say that it is quantizable), in the sense of Section 4.2. This means that there exists a way to associate, to each $f \in \mathcal{C}^\infty(M, \mathbb{C})$, a family $(\text{Op}_\hbar(f))_{\hbar \in I}$ of operators acting on Hilbert spaces $(\mathcal{H}_\hbar)_{\hbar \in I}$, and that this construction respects certain axioms. When this is the case, one can work in the reverse direction and associate to such a family $(T(\hbar))_{\hbar \in I}$ a function $f \in \mathcal{C}^\infty(M, \mathbb{C})$ such that $T(\hbar) = \text{Op}_\hbar(f) + \mathcal{O}(\hbar)$, which is called the principal symbol of $T(\hbar)$. Let $d \geq 1$ and let $(T_1(\hbar), \dots, T_d(\hbar))$ be a family of pairwise commuting semiclassical operators on M . Following the physicists, we use the following definition, which is the classical analogue of the previous one in virtue of the classical-quantum correspondence.

Definition 2.2. The *classical spectrum* of $(T_1(\hbar), \dots, T_d(\hbar))$ is the closure of the image $F(M) \subset \mathbb{R}^d$, where $F = (f_1, \dots, f_d)$ is the map of principal symbols of $T_1(\hbar), \dots, T_d(\hbar)$.

2.2. Hausdorff distance. The *Hausdorff distance* (see e.g. [5, Definition 7.3.1]) between two subsets $A \subset X$ and $B \subset X$ of a metric space (X, d) is the quantity

$$d_H^X(A, B) := \inf \{ \varepsilon > 0 \mid A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon \},$$

where for any subset S of X , and for any $\varepsilon > 0$, the set S_ε is defined as $S_\varepsilon := \cup_{s \in S} \{x \in X \mid d(s, x) \leq \varepsilon\}$. Recall that if $d_H^X(A, B) = 0$ and A, B are closed sets, then $A = B$. When $X = \mathbb{R}^d$ with its Euclidean norm, we will simply use the notation d_H for the Hausdorff distance. In the following results, convergence is meant with respect to the Hausdorff distance.

2.3. Semiclassical limits of self-adjoint operators. Let $(T_1(\hbar), \dots, T_d(\hbar))$ be a family of semiclassical self-adjoint operators on a quantizable manifold M . Let $\mathcal{S} \subset \mathbb{R}^d$ be the classical spectrum of $(T_1(\hbar), \dots, T_d(\hbar))$. Let $C_h := \text{Convex Hull}(\text{JointSpec}(T_1(\hbar), \dots, T_d(\hbar)))$.

Theorem 2.3 ([21, Theorem 8]). *Assume that for every $j \in \llbracket 1, d \rrbracket$, the principal symbol of $T_j(\hbar)$ is bounded. Then the limit of $\{C_h\}_{h \in I}$, as h tends to zero, exists and coincides with $\overline{\text{Convex Hull}(\mathcal{S})}$.*

3. MAIN THEOREM: SEMICLASSICAL LIMITS OF UNITARY OPERATORS

Let M be a connected quantizable manifold. Let $d \geq 1$ and consider a family $\mathcal{F} := (U_1(\hbar), \dots, U_d(\hbar))$ of pairwise commuting unitary semiclassical operators on M (necessarily their principal symbols are \mathbb{S}^1 -valued, as we will prove in Lemma 4.8). Let f_j be the principal symbol of $U_j(\hbar)$. Let $\mathcal{S} \subset \mathbb{T}^d$ be the classical spectrum of \mathcal{F} . For a subset A of \mathbb{T}^d we denote by $\text{Convex Hull}_{\mathbb{T}^d}(A)$ its “convex hull” in the torus (the construction of this convex hull is subtle and we carry it out in Section 5). We introduce $C_h := \text{Convex Hull}_{\mathbb{T}^d}(\text{JointSpec}(U_1(\hbar), \dots, U_d(\hbar)))$. For a complete version of the following statement, where the “genericity conditions” are spelled out, see Theorem 6.6.

Theorem 3.1. *Suppose that no f_j , $j \in \llbracket 1, d \rrbracket$, is onto, and that the images $f_j(M)$, $j \in \llbracket 1, d \rrbracket$ as well as the joint image $(f_1, \dots, f_d)(M)$ are closed. Then from $\{C_h\}_{h \in I}$, one can recover the convex hull of \mathcal{S} . Furthermore, under some generic assumptions, the limit of $\{C_h\}_{h \in I}$, as \hbar tends to zero, exists and coincides with $\text{Convex Hull}_{\mathbb{T}^d} \mathcal{S}$.*

Suppose that the closed, connected symplectic manifold M is prequantizable, i.e. the cohomology class of ω divided by 2π is integral, and endowed with a non-Hamiltonian symplectic \mathbb{T}^d -action with momentum map $\mu := (\mu_1, \dots, \mu_d) : M \rightarrow \mathbb{T}^d$ (μ always exists, by a theorem of McDuff [19]). Let \mathcal{S} be the image of μ . Theorem 3.1 implies (see details in Section 7):

Theorem 3.2. *Suppose that $(U_1(\hbar), \dots, U_d(\hbar))$ is a family of pairwise commuting unitary Berezin-Toeplitz operators on M whose principal symbols are μ_1, \dots, μ_d , and that none of the μ_j is onto. Then from the family of*

joint spectra one can recover the convex hull of \mathcal{S} , and under some generic assumptions, $\lim_{\hbar \rightarrow 0} C_\hbar = \text{Convex Hull}_{\mathbb{T}^d} \mathcal{S}$.

4. SEMICLASSICAL OPERATORS AND QUANTIZATION

4.1. Operators on Hilbert spaces. Let \mathcal{H} be a Hilbert space, with scalar product $\langle \cdot, \cdot \rangle$; we use the notation $\|\cdot\|$ for the associated norm. We will need to work with possibly unbounded linear operators acting on \mathcal{H} , hence we introduce some standard terminology (for more details, we refer the reader to standard material, as [23, Chapter VIII] or [15, Appendix 3] for instance). A linear operator acting on \mathcal{H} is the data of a linear subspace $\mathcal{D}(T) \subset \mathcal{H}$, called the domain of T , and a linear map $T : \mathcal{D}(T) \rightarrow \mathcal{H}$. Throughout the paper, $\mathcal{L}(\mathcal{H})$ will denote the set of densely defined (that is with dense domain) linear operators on \mathcal{H} . The range $\mathcal{R}(T)$ of a linear operator T is the set of all values Tu , $u \in \mathcal{D}(T)$.

We say that the operator T is *bounded* if there exists a constant $C \geq 0$ such that for every $u \in \mathcal{D}(T)$, $\|Tu\| \leq C\|u\|$. If this is the case, by a slight abuse of notation, we will write $\|T\|$ for its operator norm, defined as

$$\|T\| = \sup_{u \in \mathcal{D}(T), u \neq 0} \frac{\|Tu\|}{\|u\|}.$$

Let us recall that if T is a bounded operator, it admits a bounded extension with domain \mathcal{H} (see [15, Proposition A.3.9] for example).

If T is a densely defined linear operator acting on \mathcal{H} , its adjoint is defined as follows: let $\mathcal{D}(T^*)$ be the set of $u \in \mathcal{H}$ such that there exists $v_u \in \mathcal{H}$ satisfying: $\forall w \in \mathcal{D}(T)$, $\langle Tw, u \rangle = \langle w, v_u \rangle$. Then for $u \in \mathcal{D}(T^*)$, this v_u is unique and we set $T^*u = v_u$. This defines a linear operator acting on \mathcal{H} , with domain $\mathcal{D}(T^*)$ not necessarily dense; T^* is called the adjoint of T . A densely defined closed operator is said to be *normal* when $TT^* = T^*T$ (this equality includes the fact that the domains of these operators agree). Normal operators are of particular interest because they satisfy the spectral theorem [10, Chapter X, Theorem 4.11] which associates to the operator a spectral measure and spectral projections. Two normal operators $A, B \in \mathcal{L}(\mathcal{H})$ are said to *commute* if and only if all their spectral projections commute (cf. for instance [26, Proposition 5.27]). A densely defined operator T is said to be *self-adjoint* when $T^* = T$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *positive*, in which case we will write $T \geq 0$, when $\langle Tu, u \rangle \geq 0$ for every $u \in \mathcal{D}(T)$; if there exists some constant $c \in \mathbb{R}$ such that $T - c \text{Id} \geq 0$, then we write $T \geq c \text{Id}$.

We say that $T \in \mathcal{L}(\mathcal{H})$ is *invertible* if it admits a bounded inverse, that is a bounded operator $T^{-1} : \mathcal{R}(T) \rightarrow \mathcal{D}(T)$ such that $TT^{-1} = \text{Id}_{\mathcal{R}(T)}$ and $T^{-1}T = \text{Id}_{\mathcal{D}(T)}$. In this case, T^{-1} is unique. A bounded operator U acting on \mathcal{H} is said to be *unitary* if it is invertible and $U^{-1} = U^*$. Now, we define the spectrum $\text{Sp}(T) \subset \mathbb{C}$ of a given $T \in \mathcal{L}(\mathcal{H})$ as follows: $\lambda \in \text{Sp}(T)$ if and only if $\lambda \text{Id} - T$ is not invertible. It is standard that the spectrum of a

self-adjoint (respectively unitary) operator is a subset of \mathbb{R} (respectively the unit circle \mathbb{S}^1).

Finally, recall the following useful result about the norm of a self-adjoint operator. If A is self-adjoint, then

$$(1) \quad \sup_{\lambda \in \text{Sp}(A)} |\lambda| = \sup_{\substack{u \in \mathcal{D}(A) \\ u \neq 0}} \frac{|\langle Au, u \rangle|}{\|u\|^2} = \sup_{\substack{u \in \mathcal{D}(A) \\ u \neq 0}} \frac{\|Au\|}{\|u\|} \leq +\infty.$$

This result is standard but very often stated for bounded operators only; a concise proof can be found in [21, Section 3].

4.2. Semiclassical quantization. Let M be a connected manifold. Let \mathcal{A}_0 be a subalgebra of $\mathcal{C}^\infty(M, \mathbb{C})$ containing the constants and the compactly supported functions, and stable by complex conjugation. Assume also that if $f \in \mathcal{A}_0$ never vanishes, then $1/f$ also belongs to \mathcal{A}_0 . Let $I \subset (0, 1]$ be a set accumulating at zero. Given a bounded function $f \in \mathcal{A}_0$, its uniform norm will be denoted by $\|f\|_\infty$.

Definition 4.1. A semiclassical quantization of (M, \mathcal{A}_0) consists of a family of complex Hilbert spaces $(\mathcal{H}_h)_{h \in I}$ together with a family of \mathbb{C} -linear maps $\text{Op}_h : \mathcal{A}_0 \rightarrow \mathcal{L}(\mathcal{H}_h)$ satisfying the following properties (in the statement of which $f, g \in \mathcal{A}_0$):

- (Q1) if f and g are bounded, then the composition $\text{Op}_h(f) \text{Op}_h(g)$ is well-defined and $\|\text{Op}_h(f) \text{Op}_h(g) - \text{Op}_h(fg)\| = \mathcal{O}(\hbar)$ (*composition*);
- (Q2) for every $h \in I$, $\text{Op}_h(f)^* = \text{Op}_h(\bar{f})$ (*reality*);
- (Q3) $\text{Op}_h(1) = \text{Id}$ (*normalization*);
- (Q4) if $f \geq 0$, then there exists a constant $C > 0$ such that for every $h \in I$, $\text{Op}_h(f) \geq -C\hbar \text{Id}$, (*quasi-positivity*);
- (Q5) if $f \neq 0$ has compact support, then $\text{Op}_h(f)$ is bounded for every $h \in I$ and $\liminf_{h \rightarrow 0} \|\text{Op}_h(f)\| > 0$ (*non-degeneracy*);
- (Q6) if g has compact support, then for every $f \in \mathcal{A}_0$, $\text{Op}_h(f) \text{Op}_h(g)$ is bounded and $\|\text{Op}_h(f) \text{Op}_h(g) - \text{Op}_h(fg)\| = \mathcal{O}(\hbar)$ (*product formula*);

Definition 4.2. If such a semiclassical quantization exists, we say that M is *quantizable*.

Let us make a few comments regarding these axioms. Axioms (Q3), (Q4), (Q5) and (Q6) were introduced in [21] in order to work with self-adjoint semiclassical operators¹. We introduce axioms (Q1) and (Q2) in order to extend this setting to include operators that are not necessarily self-adjoint. One could argue that there is some redundancy between axioms (Q1) and

¹Actually, axiom (Q3) was stated in the weaker form $\|\text{Op}_h(1) - \text{Id}\| = \mathcal{O}(\hbar)$, but our formulation does not seem too restrictive, since it is still true for both Berezin-Toeplitz and pseudodifferential quantizations.

(Q6), but for the sake of clarity, we prefer to keep them both instead of stating some single axiom implying them both.

It was checked in [21] that the axioms (Q3), (Q4), (Q5) and (Q6) are satisfied by pseudodifferential and Berezin-Toeplitz operators. For pseudodifferential operators, axiom (Q1) is a consequence of the product formula for the Weyl quantization, which can be found in [11, Theorem 7.9] for example, while axiom (Q2) is given by Formula (7.3) (with $t = 1/2$) in the same reference. The fact that axioms (Q1) and (Q2) hold for Berezin-Toeplitz quantization will be checked in Lemma 7.2.

Let us now derive a few consequences of these axioms. Firstly, note that axiom (Q2) implies in particular that Op_h maps real-valued functions to self-adjoint operators. Similarly, axioms (Q1), (Q2) and (Q3) together imply that Op_h maps \mathbb{S}^1 -valued functions to “quasi-unitary” operators, that is to say operators $U_h \in \mathcal{L}(\mathcal{H})$ such that

$$\|U_h^* U_h - \text{Id}\| = \mathcal{O}(\hbar), \quad \|U_h U_h^* - \text{Id}\| = \mathcal{O}(\hbar).$$

Indeed, if f is such a function, then we have by axioms (Q2) and (Q3) that

$$\|\text{Op}_h(f)^* \text{Op}_h(f) - \text{Id}\| = \|\text{Op}_h(\bar{f}) \text{Op}_h(f) - \text{Op}_h(1)\|;$$

since f is \mathbb{S}^1 -valued, this yields

$$\|\text{Op}_h(f)^* \text{Op}_h(f) - \text{Id}\| = \|\text{Op}_h(\bar{f}) \text{Op}_h(f) - \text{Op}_h(|f|^2)\| = \mathcal{O}(\hbar)$$

thanks to axiom (Q1). Secondly, our axioms show the following.

Corollary 4.3. *If $f \in \mathcal{A}_0$ is bounded, then the operator $\text{Op}_h(f)$ is bounded and satisfies*

$$(2) \quad \|\text{Op}_h(f)\| \leq \|f\|_\infty + \mathcal{O}(\hbar).$$

Proof. Axioms (Q3) and (Q4) yield that $\text{Op}_h(|f|^2) \leq \|f\|_\infty^2 \text{Id} + \mathcal{O}(\hbar)$. Since $\text{Op}_h(|f|^2)$ is self-adjoint, this implies, by formula (1), that its norm satisfies $\|\text{Op}_h(|f|^2)\| \leq \|f\|_\infty^2 + \mathcal{O}(\hbar)$; because of axioms (Q1) and (Q2), this means that

$$\|\text{Op}_h(f)^* \text{Op}_h(f)\| \leq \|f\|_\infty^2 + \mathcal{O}(\hbar).$$

But this, in turn, yields the boundedness of $\text{Op}_h(f)$; indeed, if $u \in \mathcal{H}$ belongs to the domain of $\text{Op}_h(f)$, then we get by the Cauchy-Schwarz inequality that

$$|\langle \text{Op}_h(f)u, \text{Op}_h(f)u \rangle| = |\langle \text{Op}_h(f)^* \text{Op}_h(f)u, u \rangle| \leq \|\text{Op}_h(f)^* \text{Op}_h(f)u\| \|u\|.$$

Therefore, we obtain that $\|\text{Op}_h(f)u\| \leq \sqrt{\|\text{Op}_h(f)^* \text{Op}_h(f)\|} \|u\|$, which implies that $\text{Op}_h(f)$ is bounded and that its norm satisfies (2). \square

We state another useful corollary of our axioms regarding invertibility.

Corollary 4.4. *Let $f \in \mathcal{A}_0$ be bounded. The following are equivalent:*

- *there exists $\hbar_0 \in I$ such that for every $\hbar \leq \hbar_0$, $\text{Op}_h(f)$ is invertible and the norm of its inverse is uniformly bounded in \hbar ,*
- *there exists $c > 0$ such that $|f| \geq c$.*

Proof. Note that since f is bounded, the previous corollary yields that $\text{Op}_h(f)$ is bounded with norm smaller than $\|f\|_\infty + \mathcal{O}(\hbar)$. Assume that $\text{Op}_h(f)$ is invertible for $\hbar \leq \hbar_0$ with $\|\text{Op}_h(f)^{-1}\| \leq 1/c$ for every $\hbar \leq \hbar_0$, for some constant $c > 0$. Then from $\text{Op}_h(f)^{-1}\text{Op}_h(f) = \text{Id}$, we derive the following inequality:

$$(3) \quad \forall u \in \mathcal{H}_h \quad \|\text{Op}_h(f)u\| \geq \frac{\|u\|}{\|\text{Op}_h(f)^{-1}\|} \geq c\|u\|.$$

Let $m \in M$ and choose a compact set $\tilde{K} \subset M$ such that $|f(p) - f(m)| \leq \frac{c}{4}$ for all $p \in \tilde{K}$. Let $\chi \geq 0$ be a smooth function identically equal to one on a compact set K containing m included in the interior of \tilde{K} and with compact support contained in \tilde{K} . We claim that there exists $u_h \in \mathcal{H}_h$ of unit norm such that

$$(4) \quad u_h = \text{Op}_h(\chi)u_h + \mathcal{O}(\hbar).$$

This claim is established in Step 3 of the proof of Lemma 11 in [21], but we present a sketch of its proof for the sake of completeness. Let η be a smooth, not identically vanishing function supported on K . By axiom (Q5), there exists $\gamma > 0$ such that $\|\text{Op}_h(\eta)\| \geq \gamma$ for every $\hbar \leq \hbar_0$, so there exists some $v_h \in \mathcal{H}_h$ of norm 1 and such that $\|\text{Op}_h(\eta)v_h\| > \gamma/2$. Choose u_h as follows:

$$u_h = \frac{1}{\|\text{Op}_h(\eta)v_h\|} \text{Op}_h(\eta)v_h.$$

Thanks to axiom (Q6), we obtain

$$\text{Op}_h(\chi)u_h = \frac{1}{\|\text{Op}_h(\eta)v_h\|} \text{Op}_h(\chi\eta)v_h + \mathcal{O}(\hbar)$$

which allows us to conclude that u_h satisfies formula (4), since $\chi\eta = \eta$.

We choose such a u_h . By axiom (Q6), we get that

$$\|\text{Op}_h(\chi f)u_h - \text{Op}_h(f)\text{Op}_h(\chi)u_h\| = \mathcal{O}(\hbar).$$

Combining this estimate with the fact that u_h satisfies equation (4) yields $\|\text{Op}_h(\chi f)u_h - \text{Op}_h(f)u_h\| = \mathcal{O}(\hbar)$ and using equations (2) and (3), this gives

$$\|\chi f\|_\infty + \mathcal{O}(\hbar) \geq \|\text{Op}_h(\chi f)u_h\| \geq \|\text{Op}_h(f)u_h\| + \mathcal{O}(\hbar) \geq c + \mathcal{O}(\hbar).$$

By choosing \hbar sufficiently small, this yields $\|\chi f\|_\infty \geq c/2$. Since $0 \leq \chi \leq 1$, this means that there exists $p \in \tilde{K}$ such that $|f(p)| \geq c/2$. But by our choice of \tilde{K} , this yields $|f(m)| \geq |f(p)| - \frac{c}{4} \geq \frac{c}{4}$.

Conversely, assume that $|f| \geq c$ for some constant $c > 0$. Then $1/f$ is bounded, thus axiom (Q1) implies that $\text{Op}_h(f)\text{Op}_h\left(\frac{1}{f}\right) = \text{Id} + R_h$ where R_h is bounded with norm $\mathcal{O}(\hbar)$. By a standard result (see for instance [15, Theorem A3.30]), there exists $\hbar_1 \in I$ such that $\text{Id} + R_h$ is invertible whenever

$\hbar \leq \hbar_1$, thus for such \hbar

$$\mathrm{Op}_\hbar(f) \mathrm{Op}_\hbar\left(\frac{1}{f}\right) (\mathrm{Id} + R_\hbar)^{-1} = \mathrm{Id},$$

therefore $\mathrm{Op}_\hbar(f)$ is surjective. Similarly, there exists a bounded operator S_\hbar with norm $\mathcal{O}(\hbar)$ such that $\mathrm{Op}_\hbar\left(\frac{1}{f}\right) \mathrm{Op}_\hbar(f) = \mathrm{Id} + S_\hbar$ and there exists $\hbar_2 \in I$ such that for every $\hbar \leq \hbar_2$, $\mathrm{Id} + S_\hbar$ is invertible, so

$$(\mathrm{Id} + S_\hbar)^{-1} \mathrm{Op}_\hbar\left(\frac{1}{f}\right) \mathrm{Op}_\hbar(f) = \mathrm{Id}$$

and hence $\mathrm{Op}_\hbar(f)$ is injective. Consequently, $\mathrm{Op}_\hbar(f)$ is bijective for every $\hbar \leq \hbar_0 := \min(\hbar_1, \hbar_2)$. Since $\mathrm{Op}_\hbar(f)$ is a bounded operator, the inverse mapping theorem [23, Theorem III.11] implies that it is invertible for every $\hbar \leq \hbar_0$. It remains to show that the norm of its inverse is uniformly bounded in \hbar . For this we notice that, by Corollary 4.3, $\mathrm{Op}_\hbar(1/f)$ is bounded since $1/f$ is bounded, and we have the inequality

$$\|\mathrm{Op}_\hbar(f)^{-1}\| \leq \|\mathrm{Id} + S_\hbar\|^{-1} \left\| \mathrm{Op}_\hbar\left(\frac{1}{f}\right) \right\| \leq \left\| \frac{1}{f} \right\|_\infty + \mathcal{O}(\hbar).$$

This implies that the norm of $\mathrm{Op}_\hbar(f)^{-1}$ is uniformly bounded in \hbar . \square

Remark 4.5. Note that as a byproduct of the proof of the second point of the corollary, we have that if f is bounded and $|f| \geq c$ for some $c > 0$, then $\left\| \mathrm{Op}_\hbar(f)^{-1} - \mathrm{Op}_\hbar\left(\frac{1}{f}\right) \right\| = \mathcal{O}(\hbar)$. \otimes

4.3. Semiclassical operators. We now introduce an algebra \mathcal{A}_I whose elements are families $f_I = (f_\hbar)_{\hbar \in I}$ of functions in \mathcal{A}_0 of the form $f_\hbar = f_0 + \hbar f_{1,\hbar}$ with $f_0 \in \mathcal{A}_0$ and where the family $(f_{1,\hbar})_{\hbar \in I}$ is uniformly bounded in \hbar and supported in a compact subset of M which does not depend on \hbar . If f_0 is also compactly supported, we say that f_I is compactly supported. We have a map

$$\mathrm{Op} : \mathcal{A}_I \rightarrow \prod_{\hbar \in I} \mathcal{L}(\mathcal{H}_\hbar), f_I = (f_\hbar)_{\hbar \in I} \mapsto (\mathrm{Op}_\hbar(f_\hbar))_{\hbar \in I}.$$

Definition 4.6. A *semiclassical operator* is any element of the image of this map. We denote by $\Psi := \mathrm{Op}(\mathcal{A}_I)$ the set of semiclassical operators.

We want to define a map $\sigma : \Psi \rightarrow \mathcal{A}_0$ which associates to $\mathrm{Op}_\hbar(f_I)$ the function $f_0 \in \mathcal{A}_0$. However, we need to check that the latter is unique.

Lemma 4.7. *The map σ is well-defined. Given $T = (T_\hbar)_{\hbar \in I} \in \Psi$, we call $\sigma(T)$ the principal symbol of T .*

Proof. This proof already appeared in [21, Section 4] but we recall it here for the sake of completeness. Let $f_I \in \mathcal{A}_I$ be such that $\text{Op}(f_I) = 0$. Since the family $(f_{1,\hbar})_{\hbar \in I}$ is uniformly bounded in \hbar , we deduce from Corollary 4.3 that

$$(5) \quad \|\text{Op}_\hbar(f_\hbar) - \text{Op}_\hbar(f_0)\| = \mathcal{O}(\hbar).$$

Let χ be any compactly supported smooth function. Using the previous estimate and axiom (Q6), we obtain that

$$\|\text{Op}_\hbar(f_\hbar)\text{Op}_\hbar(\chi) - \text{Op}_\hbar(f_\hbar\chi)\| = \mathcal{O}(\hbar),$$

hence $\|\text{Op}_\hbar(f_\hbar\chi)\| = \mathcal{O}(\hbar)$. Consequently, applying Equation (5) to $f_\hbar\chi$ yields the equality $\|\text{Op}_\hbar(f_0\chi)\| = \mathcal{O}(\hbar)$. Therefore, by axiom (Q5), we conclude that $f_0\chi = 0$. Since χ was arbitrary, this means that $f_0 = 0$. \square

By axiom (Q3), the principal symbol of the identity is $\sigma(\text{Id}) = 1$. Axiom (Q2) implies that the principal symbol of a self-adjoint semiclassical operator is real-valued. We can also draw conclusions about the principal symbol of a unitary operator.

Lemma 4.8. *The principal symbol of a unitary semiclassical operator is \mathbb{S}^1 -valued.*

Proof. Let U_\hbar be a unitary semiclassical operator. Since we are only interested in the principal symbol, we can assume that $U_\hbar = \text{Op}_\hbar(f)$ for some $f \in \mathcal{A}_0$. Let $m \in M$ and let $\chi \geq 0$ be a smooth compactly supported function such that $\chi(m) = 1$. By axiom (Q6), we get that

$$(6) \quad \|\text{Op}_\hbar(\chi^2|f|^2) - \text{Op}_\hbar(\chi\bar{f})\text{Op}_\hbar(\chi f)\| = \mathcal{O}(\hbar).$$

But, still because of axiom (Q6), we have that $\|\text{Op}_\hbar(\chi f) - \text{Op}_\hbar(f)\text{Op}_\hbar(\chi)\| = \mathcal{O}(\hbar)$, which yields thanks to Corollary 4.3 applied to $\chi\bar{f}$:

$$\|\text{Op}_\hbar(\chi\bar{f})\text{Op}_\hbar(\chi f) - \text{Op}_\hbar(\chi\bar{f})\text{Op}_\hbar(f)\text{Op}_\hbar(\chi)\| = \mathcal{O}(\hbar).$$

Therefore we obtain by using (6) and the triangle inequality:

$$\|\text{Op}_\hbar(\chi^2|f|^2) - \text{Op}_\hbar(\chi\bar{f})\text{Op}_\hbar(f)\text{Op}_\hbar(\chi)\| = \mathcal{O}(\hbar).$$

By iterating the same method, we eventually get

$$\|\text{Op}_\hbar(\chi^2|f|^2) - \text{Op}_\hbar(\chi)\text{Op}_\hbar(\bar{f})\text{Op}_\hbar(f)\text{Op}_\hbar(\chi)\| = \mathcal{O}(\hbar).$$

Now, using axiom (Q2) and the fact that $\text{Op}_\hbar(f)$ is unitary, this yields $\|\text{Op}_\hbar(\chi^2|f|^2) - \text{Op}_\hbar(\chi)^2\| = \mathcal{O}(\hbar)$. Finally, thanks to axiom (Q6) and the linearity of Op_\hbar , we infer from this equality that

$$\|\text{Op}_\hbar(\chi^2(|f|^2 - 1))\| = \mathcal{O}(\hbar),$$

thus as a consequence of axiom (Q5) we have that $\chi^2(|f|^2 - 1) = 0$, hence $|f(m)|^2 = 1$. \square

Let us state a final useful remark regarding semiclassical operators. Let $T_h \in \Psi$ be a semiclassical operator with bounded principal symbol f , and such that $|f| \geq c$ for some $c > 0$. Then as a consequence of Corollary 4.4, T_h is invertible for h sufficiently small. Indeed, $\text{Op}_h(f)$ is invertible and $T_h = \text{Op}_h(f) + \mathcal{O}(h)$; thus our claim comes from an application of [15, Theorem A.3.30].

5. THE CONVEX HULL OF A SUBSET OF \mathbb{T}^d

In this section we propose a way to define the convex hull of a subset of the torus \mathbb{T}^d . We believe that the definition of toric convex hull that we introduce here could be of interest in other settings, for instance in computational geometry; see [13] where a first definition was proposed, but the convex hulls constructed using it were too large for practical applications.

5.1. Preliminaries about \mathbb{T}^d . We consider $\mathbb{T}^d = (\mathbb{S}^1)^d$ as the product of d copies of the unit circle. If z belongs to the unit circle, we will denote by $\arg(z)$ its argument in $(-\pi, \pi]$. Furthermore, we will also denote by $\arg : \mathbb{T}^d = (\mathbb{S}^1)^d \rightarrow (-\pi, \pi]^d$ the function assigning its argument to each component of $z \in \mathbb{T}^d$: $\arg(z_1, \dots, z_d) = (\arg(z_1), \dots, \arg(z_d))$. Similarly, we will consider the function

$$\exp : \mathbb{C}^d \rightarrow \mathbb{C}^d, \quad (w_1, \dots, w_d) \mapsto (\exp(w_1), \dots, \exp(w_d)).$$

We endow \mathbb{T}^d with the following distance: for $z, w \in \mathbb{T}^d$

$$d^{\mathbb{T}^d}(z, w) = \min_{\theta \in (2\pi\mathbb{Z})^d} \|\arg(z) - \arg(w) + \theta\|_{\mathbb{R}^d}.$$

The Hausdorff distance induced by this distance will be denoted by $d_H^{\mathbb{T}^d}$.

5.2. Multiplication in \mathbb{T}^d . Let $a = (a_1, \dots, a_d), b = (b_1, \dots, b_d)$ be two points in $\mathbb{T}^d = (\mathbb{S}^1)^d$. Then we use the following notation for the product of a and b in \mathbb{T}^d : $a \cdot b = (a_1 b_1, \dots, a_d b_d)$. Now, given a subset E of the torus \mathbb{T}^d and a point $a \in \mathbb{T}^d$, we define the set $a \cdot E$ as the set of all points of the form $a \cdot z$, $z \in E$. Moreover, we use the notation $a^{-1} \in \mathbb{T}^d$ to denote the point $(a_1^{-1}, \dots, a_d^{-1})$. An easy consequence of our choice of distance on \mathbb{T}^d is that the associated Hausdorff distance $d_H^{\mathbb{T}^d}$ is multiplication invariant.

Lemma 5.1. *Let $E, F \subset \mathbb{T}^d$. Then $d_H^{\mathbb{T}^d}(a \cdot E, a \cdot F) = d_H^{\mathbb{T}^d}(E, F)$ for every $a \in \mathbb{T}^d$.*

5.3. Convex hulls for simple subsets of \mathbb{T}^d . If we could lift everything to \mathbb{R}^d without any trouble, we would define the convex hull of a subset E of \mathbb{T}^d as the projection of the convex hull of its lift. This naive idea cannot be used in general, but can be adapted for what we will call simple subsets (see the definitions below), whose image under \arg is contained in $(-\pi, \pi)^d$. For instance, it works as is if E is simple and connected. However, if we drop connectedness, which is a natural thing to do since one often wants to

compute the convex hull of a collection of points, then choices are involved, and defining the convex hull is subtle.

Definition 5.2. A subset $E \subset \mathbb{T}^d$ is called *very simple* if for every point $(z_1, \dots, z_d) \in E$ and for all $j \in \llbracket 1, d \rrbracket$, $z_j \neq -1$.

For $j \in \llbracket 1, d \rrbracket$, let $p_j : \mathbb{T}^d = (\mathbb{S}^1)^d \rightarrow \mathbb{S}^1$ be the projection on the j -th factor.

Definition 5.3. A subset $E \subset \mathbb{T}^d$ is called *simple* if no $p_{j|E}$ is onto.

Remark 5.4. If E is simple, there exists $a \in \mathbb{T}^d$ such that $a \cdot E$ is very simple. A set consisting of a finite number of points is always simple. \otimes

Recall that for a subset E of the torus \mathbb{T}^d and a point $a \in \mathbb{T}^d$, we define the set $a \cdot E$ as the set of points of the form $a \cdot z$, $z \in E$.

Lemma 5.5. *Let $E \subset \mathbb{T}^d$ be simple and compact, with finitely many connected components E_1, \dots, E_n . Let $b, c \in \mathbb{T}^d$ be such that both $b \cdot E$ and $c \cdot E$ are very simple. Then for every $j \in \llbracket 1, N \rrbracket$, there exists a constant $\theta_j^{(b,c)} \in (2\pi\mathbb{Z})^d$ such that for all $z \in E_j$,*

$$\arg(c \cdot z) = \arg(b \cdot z) + \arg(c \cdot b^{-1}) + \theta_j^{(b,c)}.$$

We call $\theta_j^{(b,c)}$ the phase shift of E_j with respect to (b, c) .

Proof. Let $z \in E$; then $c \cdot z = (c \cdot b^{-1}) \cdot (b \cdot z)$, therefore

$$\arg(c \cdot z) = \arg(b \cdot z) + \arg(c \cdot b^{-1}) + \theta(z)$$

for some $\theta(z) \in (2\pi\mathbb{Z})^d$. But the function $z \mapsto \arg(c \cdot z) - \arg(b \cdot z)$ is continuous, since $b \cdot E$ and $c \cdot E$ are very simple; indeed, if a compact set $H \subset \mathbb{T}^d$ is very simple, then H is contained in some compact subset K of $(\mathbb{S}^1 \setminus \{-1\})^d$, and the function $\arg : K \rightarrow (-\pi, \pi)^d$ is continuous. Hence the same holds for $z \mapsto \theta(z)$, and thus $\theta(z) = \theta(w)$ whenever z and w belong to the same connected component of E . Consequently, for every $j \in \llbracket 1, d \rrbracket$, there exists a constant $\theta_j^{(b,c)} \in (2\pi\mathbb{Z})^d$ such that for every $z \in E_j$, $\theta(z) = \theta_j^{(b,c)}$, which was to be proved. \square

Given a compact subset I of \mathbb{R} , we use the notation $\text{diam}(I)$ for the diameter of I : $\text{diam}(I) = \max \{|x - y|, \ x, y \in I\}$, with the convention that $\text{diam}(\emptyset) = 0$. Furthermore, let η_j , $1 \leq j \leq d$ denote the natural projection $\mathbb{R}^d \rightarrow \mathbb{R}$, $(x_1, \dots, x_d) \mapsto x_j$.

Lemma 5.6. *Let $E \subset \mathbb{T}^d$ be simple and compact, with finitely many connected components. Then there exists $b \in \mathbb{T}^d$ such that $b \cdot E$ is very simple and*

$$\forall j \in \llbracket 1, d \rrbracket, \quad \text{diam}(\eta_j(\arg(b \cdot E))) = \min \Lambda_j$$

where

$$\Lambda_j = \left\{ \text{diam}(\eta_j(\arg(c \cdot E))) \mid c \in \mathbb{T}^d, c \cdot E \text{ very simple} \right\}.$$

We say that such a point $b \in \mathbb{T}^d$ is admissible for E .

Proof. We start by proving that the sets Λ_j do admit minima. Let $j \in \llbracket 1, d \rrbracket$; since E is simple, Λ_j is not empty. We will prove that the set Λ_j consists of a finite number of values, which will yield the existence of its minimum. We now make the following simple observation: if $G = \arg(c \cdot E) \subset (-\pi, \pi]^d$ with $c \cdot E$ very simple is the image of $F = \arg(b \cdot E) \subset (-\pi, \pi]^d$, with $b \cdot E$ very simple, by a translation, then $\text{diam}(\eta_j(F)) = \text{diam}(\eta_j(G))$.

Let $\theta_1^{(b,c)}, \dots, \theta_N^{(b,c)}$ be the phase shifts of E_1, \dots, E_N with respect to (b, c) as introduced in Lemma 5.5. If G is not the image of F by a translation, then necessarily there exists $i \neq k \in \llbracket 1, N \rrbracket$ such that $\theta_i^{(b,c)} \neq \theta_k^{(b,c)}$. But each $\theta_i^{(b,c)}$ is an element of $(2\pi\mathbb{Z})^d \cap [-2\pi, 2\pi]^d$, hence we can only get a finite number of different values for $\theta_i^{(b,c)}$ by changing b and c . Consequently, there is only a finite number of ways to make G not be the image of F by a translation, hence Λ_j is finite.

It remains to prove that there exists a common $b \in \mathbb{T}^d$ minimizing all the Λ_j , $1 \leq j \leq d$. If $d = 1$, this is obvious, thus let us assume that $d \geq 2$. Obviously we can pick some $b \in \mathbb{T}^d$ which is a minimizer for Λ_1 . Now let $j \in \llbracket 1, d-1 \rrbracket$ and assume that we have found $b^j \in \mathbb{T}^d$ such that

$$\forall i \in \llbracket 1, j \rrbracket, \quad \text{diam}(\eta_i(\arg(b^j \cdot E))) = \min \Lambda_i.$$

Consider the set $C_j \subset \mathbb{T}^d$ of points of the form $(1, \tilde{c}) \cdot b_j$, $\tilde{c} \in \mathbb{T}^{d-j}$. Then clearly, for every $c \in C_j$ with $c \cdot E$ very simple,

$$\forall i \in \llbracket 1, j \rrbracket, \quad \text{diam}(\eta_i(\arg(c \cdot E))) = \min \Lambda_i.$$

Now, let $\Xi_{j+1} = \{\text{diam}(\eta_{j+1}(\arg(c \cdot E))), c \in C_j, c \cdot E \text{ very simple}\}$. We want to prove that $\min \Xi_{j+1} = \min \Lambda_{j+1}$; this follows from the fact that the map

$$\varphi_j : \mathbb{T}^j \times C_j \rightarrow \mathbb{T}^d, \quad (a, (1, \tilde{c}) \cdot b_j) \mapsto (a, \tilde{c}) \cdot b_j$$

is a bijection satisfying $\eta_{j+1}(\arg(\varphi_j(a, c) \cdot z)) = \eta_{j+1}(\arg(c \cdot z))$ for every $(a, c) \in \mathbb{T}^j \times C_j$ and $z \in E$. We conclude by (finite) induction. \square

We would now like to define the convex hull of a subset $E \subset \mathbb{T}^d$ which is simple, compact, and has finitely many connected components, as the set

$$b^{-1} \cdot \exp(i \text{Convex Hull}(\arg(b \cdot E)))$$

where $b \in \mathbb{T}^d$ is given by Lemma 5.6 and where for any set $F \subset \mathbb{R}^d$, we define $\exp(iF) = \{\exp(i\theta), \theta \in F\}$. The problem is that this point $b \in \mathbb{T}^d$ is, in general, far from being unique. Hence, in order to use this definition, we would need this set to not depend on the choice of b . But it can depend on

this choice if E displays some symmetries; this is why we use the following definition.

Definition 5.7. Let $E \subset \mathbb{T}^d$ be simple, compact, and with finitely many connected components E_1, \dots, E_N .

- (1) if for every $b, c \in \mathbb{T}^d$ which are admissible for E , all the phase shifts $\theta_1^{(b,c)}, \dots, \theta_N^{(b,c)}$ are equal, then

$$\text{Convex Hull}_{\mathbb{T}^d}(E) := b^{-1} \cdot \exp(i \text{Convex Hull}(\arg(b \cdot E)))$$

for any $b \in \mathbb{T}^d$ which is admissible for E ,

- (2) otherwise, $\text{Convex Hull}_{\mathbb{T}^d}(E) := \mathbb{T}^d$.

Remark 5.8. This definition implies that if $E \subset \mathbb{T}^d$ is simple, compact and connected, its convex hull is simply defined as

$$\text{Convex Hull}_{\mathbb{T}^d}(E) := b^{-1} \cdot \exp(i \text{Convex Hull}(\arg(b \cdot E)))$$

for any $b \in \mathbb{T}^d$ such that $b \cdot E$ is very simple. \oslash

Definition 5.7 makes sense because of the following lemma.

Lemma 5.9. *Let $b, c \in \mathbb{T}^d$ be two admissible points such that the equality $\theta_1^{(b,c)} = \dots = \theta_N^{(b,c)}$ holds. Then*

$$c^{-1} \cdot \exp(i \text{Convex Hull}(\arg(c \cdot E))) = b^{-1} \cdot \exp(i \text{Convex Hull}(\arg(b \cdot E))).$$

Proof. Because of the assumption, we have that for every $z \in E$, the equality $\arg(c \cdot z) = \arg(b \cdot z) + \arg(c \cdot b^{-1}) + \theta$ holds, where θ is the common value of the $\theta_j^{(b,c)}$. Hence

$$\text{Convex Hull}(\arg(c \cdot E)) = \arg(c \cdot b^{-1}) + \theta + \text{Convex Hull}(\arg(b \cdot E))$$

which implies, since θ belongs to $(2\pi\mathbb{Z})^d$, that

$$\exp(i \text{Convex Hull}(\arg(c \cdot E))) = c \cdot b^{-1} \cdot \exp(i \text{Convex Hull}(\arg(b \cdot E)))$$

and the result follows. \square

Definition 5.10. A simple compact subset $E \subset \mathbb{T}^d$ with finitely many connected components and satisfying the first condition in the above definition will be called *generic*.

This terminology makes sense because such sets are, indeed, generic in the following sense.

Lemma 5.11. *Let $E \subset \mathbb{T}^d$ be simple, compact, with finitely many connected components, and such that there exists $b, c \in \mathbb{T}^d$ admissible such that not all the phase shifts $\theta_j^{(b,c)}$ are equal. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_0$, there exists a compact simple subset $E_\varepsilon \subset \mathbb{T}^d$ such that E_ε is generic and $d_H^{\mathbb{T}^d}(E, E_\varepsilon) \leq \varepsilon$.*

This result, of which we will give a proof later, is a corollary of the next two lemmas. The first one gives a necessary condition for a set to be non generic.

Lemma 5.12. *Let $E = \{z_1, \dots, z_N\} \subset \mathbb{T}^d$ be such that there exists $b, c \in \mathbb{T}^d$ admissible such that not all the phase shifts $\theta_j^{(b,c)}$ are equal. Then*

(1) *either there exist $p, q, r, s \in \llbracket 1, N \rrbracket$, $j \in \llbracket 1, d \rrbracket$ and $\theta \in 2\pi\mathbb{Z}$ such that $\{p, q\} \neq \{r, s\}$ and*

$$|\eta_j(\arg(b \cdot z_r)) - \eta_j(\arg(b \cdot z_s)) + \theta| = |\eta_j(\arg(b \cdot z_p)) - \eta_j(\arg(b \cdot z_q))|,$$

(2) *or there exist $p, q \in \llbracket 1, N \rrbracket$ and $j \in \llbracket 1, d \rrbracket$ such that*

$$|\eta_j(\arg(b \cdot z_p)) - \eta_j(\arg(b \cdot z_q))| = \pi.$$

It would be interesting to give a simpler characterization of non generic sets. An example of non generic set when $d = 1$ is E consisting of a finite number of points uniformly distributed on \mathbb{S}^1 , but there are also sets with weaker symmetries which are not generic, for example

$$E = \{\exp(i\phi_1), \exp(i\phi_2), \exp(i\phi_3)\} \subset \mathbb{S}^1$$

where $\phi_3 = \pi + (\phi_1 + \phi_2)/2$ (see Figure 1).

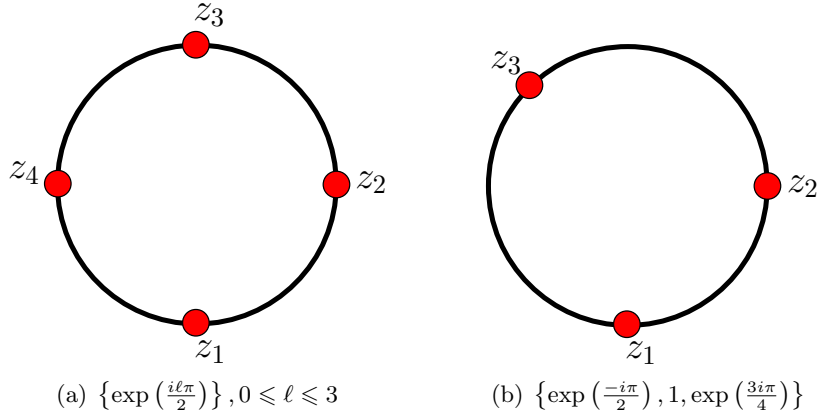


FIGURE 1. Two examples of non generic subsets of \mathbb{S}^1 .

Proof. Firstly, note that the existence of the pair (b, c) satisfying the assumptions of the lemma implies that $N > 1$. Moreover, replacing E by $b \cdot E$ and c by $c \cdot b^{-1}$ if necessary, we can assume that $b = 1$. To simplify the notation, we will set $\theta_\ell := \theta_\ell^{(1,c)}$, $1 \leq \ell \leq N$.

Let us start with some considerations for fixed $j \in \llbracket 1, d \rrbracket$. Let $p, q \in \llbracket 1, N \rrbracket$ be such that $\text{diam}(\eta_j(\arg(E))) = |\eta_j(\arg(z_p)) - \eta_j(\arg(z_q))|$. If there exist $r, s \in \llbracket 1, N \rrbracket$ with $\{p, q\} \neq \{r, s\}$ such that this diameter is also equal to $|\eta_j(\arg(z_r)) - \eta_j(\arg(z_s))|$, then we are done, because the property (1) in

the statement is satisfied with $\theta = 0$. So from now on we assume that it is not the case. We choose indices $r, s \in \llbracket 1, N \rrbracket$ such that

$$\text{diam}(\eta_j(\arg(c \cdot E))) = |\eta_j(\arg(c \cdot z_r)) - \eta_j(\arg(c \cdot z_s))|.$$

Since 1 and c are admissible, the equality

$$|\eta_j(\arg(z_p)) - \eta_j(\arg(z_q))| = |\eta_j(\arg(c \cdot z_r)) - \eta_j(\arg(c \cdot z_s))|$$

holds; it can be rewritten as

$$|\eta_j(\arg(z_p)) - \eta_j(\arg(z_q))| = |\eta_j(\arg(z_r)) - \eta_j(\arg(z_s)) + \eta_j(\theta_r) - \eta_j(\theta_s)|.$$

If $\{p, q\} \neq \{r, s\}$, then we are done again, because property (1) holds with $\theta = \eta_j(\theta_r) - \eta_j(\theta_s)$. If $\{p, q\} = \{r, s\}$ and $\eta_j(\theta_p) \neq \eta_j(\theta_q)$, then we are also done. Indeed, this means that

$$|\eta_j(\arg(z_p)) - \eta_j(\arg(z_q))| = |\eta_j(\arg(z_p)) - \eta_j(\arg(z_q)) + \theta|,$$

where $\theta = \pm 2\pi$. Assuming for instance that $\eta_j(\arg(z_p)) > \eta_j(\arg(z_q))$, this yields $2(\eta_j(\arg(z_p)) - \eta_j(\arg(z_q))) = \pm 2\pi$.

Therefore, let us consider the case where $\{p, q\} = \{r, s\}$ and $\eta_j(\theta_p) = \eta_j(\theta_q) = \mu$; we will call this case the *exceptional case*. Exchanging the roles of p and q if necessary, we can assume that $\eta_j(\arg(z_p)) > \eta_j(\arg(z_q))$. Then for every $\ell \notin \{p, q\}$, we have that

$$(7) \quad \eta_j(\arg(z_q)) < \eta_j(\arg(z_\ell)) < \eta_j(\arg(z_p)).$$

But we also know that $\eta_j(\arg(c \cdot z_p)) > \eta_j(\arg(c \cdot z_q))$, because

$$\eta_j(\arg(c \cdot z_p)) = \eta_j(\arg(z_p)) + \eta_j(\arg(c)) + \mu$$

and

$$\eta_j(\arg(c \cdot z_q)) = \eta_j(\arg(z_q)) + \eta_j(\arg(c)) + \mu.$$

Therefore, we also have for every $\ell \notin \{p, q\}$ the inequality

$$\eta_j(\arg(c \cdot z_q)) < \eta_j(\arg(c \cdot z_\ell)) < \eta_j(\arg(c \cdot z_p)),$$

which implies that

$$\eta_j(\arg(z_q)) + \mu < \eta_j(\arg(z_\ell)) + \eta_j(\theta_\ell) < \eta_j(\arg(c \cdot z_p)) + \mu.$$

Combining this with inequality (7), we get that for every $\ell \in \llbracket 1, N \rrbracket$, the equality $\eta_j(\theta_\ell) = \mu$ holds.

Let us sum up the situation. If for some $j \in \llbracket 1, d \rrbracket$, we are not in the exceptional case, then we are done. But there must exist such a j , because otherwise we would have that $\eta_j(\theta_1) = \dots = \eta_j(\theta_N)$ for every j , that is to say $\theta_1 = \dots = \theta_N$.

□

Lemma 5.13. *Let E be a compact simple subset of \mathbb{T}^d , with finitely many connected components E_1, \dots, E_N . Assume that for every $j \in \llbracket 1, d \rrbracket$ and*

$p \in \llbracket 1, N \rrbracket$, there exists a unique point $z_{j,p}^-$ (respectively $z_{j,p}^+$) such that for every $b \in \mathbb{T}^d$ with $b \cdot E$ very simple,

$$\min_{z \in E_p} \eta_j(\arg(b \cdot z)) = \eta_j(\arg(b \cdot z_{j,p}^-))$$

(respectively $\max_{z \in E_p} \eta_j(\arg(b \cdot z)) = \eta_j(\arg(b \cdot z_{j,p}^+))$). If E satisfies the assumptions of Lemma 5.11, then the set $F = \{z_{1,1}^-, z_{1,1}^+, \dots, z_{d,N}^-, z_{d,N}^+\}$ satisfies the assumptions of Lemma 5.12.

Proof. The result can be deduced from the following observation: for every $j \in \llbracket 1, d \rrbracket$ and $p \in \llbracket 1, N \rrbracket$, and for every $b \in \mathbb{T}^d$ such that $b \cdot E$ is very simple, $\text{diam}(\eta_j(\arg(b \cdot E))) = \text{diam}(\eta_j(\arg(b \cdot F)))$. \square

Proof of Lemma 5.11. Firstly, we can slightly modify the connected components of E in order to get an ε -close set \tilde{E}_ε satisfying the assumption in the previous lemma (see Figure 2), because if this assumption is true for one b such that $b \cdot E$ is very simple, it is true for all such b . To this new set \tilde{E}_ε , we then associate a set $F_\varepsilon = \{z_{1,1}^-(\varepsilon), z_{1,1}^+(\varepsilon), \dots, z_{d,N}^-(\varepsilon), z_{d,N}^+(\varepsilon)\}$ as in this lemma. Then equalities as in Lemma 5.12 occur for F_ε for a certain number of couples (b, c) of admissible points. But recall that there is only a finite number of different values of $(\theta_1^{(b,c)}, \dots, \theta_N^{(b,c)})$ that we can obtain by changing (b, c) . Hence, given $\varepsilon > 0$ small enough, by performing small perturbations of the connected components of \tilde{E}_ε around the points $z_{i,j}^\pm(\varepsilon)$ of F_ε , we can construct a set E_ε which is ε -close to \tilde{E}_ε with respect to the Hausdorff distance and such that no equality as in Lemma 5.12 ever occurs, which means that E_ε is generic. \square

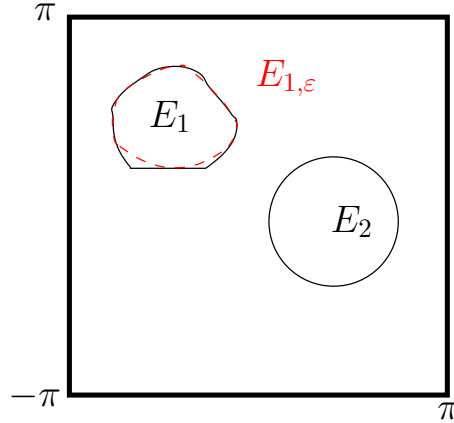


FIGURE 2. Approximating E by a set satisfying the assumptions of Lemma 5.13.

Before concluding this section, we note that the convex hull thus constructed is compatible with rotations, in the sense that if $E \subset \mathbb{T}^d$ is as in

Definition 5.7, then $\text{Convex Hull}_{\mathbb{T}^d}(b \cdot E) = b \cdot \text{Convex Hull}_{\mathbb{T}^d}(E)$ for every $b \in \mathbb{T}^d$. Indeed, if $c \in \mathbb{T}^d$ is admissible for $c \cdot E$, then $c \cdot b^{-1}$ is admissible for $b \cdot E$, and, if $C = \text{Convex Hull}_{\mathbb{T}^d}(b \cdot E)$, then

$$C = (c \cdot b^{-1})^{-1} \cdot \exp \left(i \text{Convex Hull}(\arg(c \cdot b^{-1} \cdot b \cdot E)) \right),$$

which yields

$$C = b \cdot c^{-1} \cdot \exp \left(i \text{Convex Hull}(\arg(c \cdot E)) \right) = b \cdot \text{Convex Hull}_{\mathbb{T}^d}(b \cdot E).$$

Remark 5.14. We will not give a definition of the convex hull of a general subset of the torus, since we will always keep these assumptions of compactness and finite number of connected components. However, our definition allows us to handle, in particular, compact connected subsets and sets consisting of a finite number of points. Back to our initial problem, the former corresponds to the closed image of a joint principal symbol, while the latter corresponds to the joint spectrum of a family of pairwise commuting operators acting on finite-dimensional spaces. Moreover, computing the convex hull of a finite number of points on tori seems to be of interest in computational geometry [13]. \oslash

5.4. Convex hull for compact, connected subsets of \mathbb{T}^d . We turn to the definition of the convex hull for compact connected non necessarily simple subsets.

Lemma 5.15. *Let E be a compact connected subset of \mathbb{T}^d . Assume that there exists a sequence $(E_n)_{n \geq 1}$ of compact connected very simple subsets such that*

- (1) $E_n \xrightarrow{n \rightarrow \infty} E$ with respect to the Hausdorff distance,
- (2) $E_n \subset E_{n+1}$,
- (3) $d_H^{\mathbb{T}^d}(E_n, E_{n+1}) \leq \frac{1}{2^n} \min \left(1, d \left(\arg(E_n), \partial \left([-\pi, \pi]^d \right) \right) \right)$.

We call such a sequence a very simple approximation of E . Then there exists a compact subset $C \subset \mathbb{T}^d$ such that the sequence $(C_n)_{n \geq 1}$ of subsets of \mathbb{T}^d defined by $C_n = \text{Convex Hull}_{\mathbb{T}^d}(E_n)$ converges to C for the Hausdorff distance topology.

Before proving this result, we state the following useful lemma. It is a standard exercise to show that in \mathbb{R}^d , taking the convex hull is a 1-Lipschitz operation for the Hausdorff distance; it turns out that the same does not hold in general for simple subsets of \mathbb{T}^d , see Figure 3 for a counterexample. However, the following weaker version of this property holds.

Lemma 5.16. *Let $E, F \subset \mathbb{T}^d$ be compact connected very simple subsets such that $d_H^{\mathbb{T}^d}(E, F) \leq \frac{1}{2} d \left(\arg(F), \partial \left([-\pi, \pi]^d \right) \right)$. Then*

$$d_H^{\mathbb{T}^d}(\text{Convex Hull}_{\mathbb{T}^d}(E), \text{Convex Hull}_{\mathbb{T}^d}(F)) \leq d_H^{\mathbb{T}^d}(E, F).$$

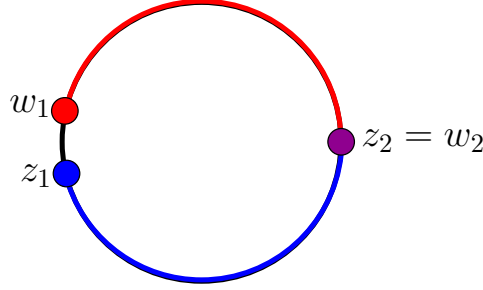


FIGURE 3. Two very simple subsets $E = \{z_1, z_2\}$ and $F = \{w_1, w_2\}$ of \mathbb{S}^1 whose convex hulls (in blue, the convex hull of E , in red, the convex hull of F) are at Hausdorff distance greater than the Hausdorff distance between E and F .

Proof. Before starting the proof, we recall that, because of Definition 5.7 and the remark following it, $\text{Convex Hull}_{\mathbb{T}^d}(E) = \exp(i(\text{Convex Hull}_{\mathbb{T}^d}(\arg(E))))$ and similarly for F .

Let $z \in \text{Convex Hull}_{\mathbb{T}^d}(F)$; there exists $\theta \in \text{Convex Hull}(\arg(F))$ such that $z = \exp(i\theta)$. Thus θ can be written as a finite linear combination of elements of $\arg(F)$: there exist $\alpha_1, \dots, \alpha_m \in [0, 1]$ and $\theta^1, \dots, \theta^d \in \arg(F)$ such that $\sum_{\ell=1}^m \alpha_\ell = 1$ and $\theta = \sum_{\ell=1}^m \alpha_\ell \theta^\ell$ (here we use superscripts to avoid confusion with the components of elements of \mathbb{R}^d). For $1 \leq \ell \leq m$, let $z^\ell = \exp(i\theta^\ell) \in F$. Fix $1 < \gamma < 2$; there exists $w^1, \dots, w^m \in E$ such that for $1 \leq \ell \leq m$

$$(8) \quad d^{\mathbb{T}^d}(z^\ell, w^\ell) \leq \gamma d_H^{\mathbb{T}^d}(E, F) \leq \gamma \delta,$$

where $\delta = \frac{1}{2}d(\arg(F), \partial([- \pi, \pi]^d))$. Consider

$$w = \exp\left(i \sum_{\ell=1}^m \alpha_\ell \arg(w^\ell)\right) \in \text{Convex Hull}_{\mathbb{T}^d}(E).$$

For $\ell \in [1, m]$, choose a non-zero $\phi^\ell \in (2\pi\mathbb{Z})^d$; then $\arg(w^\ell) - \phi^\ell$ does not belong to $[-\pi, \pi]^d$, thus

$$\left\| \arg(z^\ell) - \arg(w^\ell) + \phi^\ell \right\|_{\mathbb{R}^d} \geq d\left(\arg(F), \partial([- \pi, \pi]^d)\right),$$

which implies, by the choice of γ , that

$$\left\| \arg(z^\ell) - \arg(w^\ell) + \phi^\ell \right\|_{\mathbb{R}^d} \geq \gamma \delta \geq \left\| \arg(z^\ell) - \arg(w^\ell) \right\|_{\mathbb{R}^d}.$$

Therefore $d^{\mathbb{T}^d}(z^\ell, w^\ell) = \|\theta^\ell - \arg(w^\ell)\|_{\mathbb{R}^d}$. Combining this equality with the fact that

$$d^{\mathbb{T}^d}(z, w) \leq \left\| \sum_{\ell=1}^m \alpha_\ell \left(\arg(w^\ell) - \theta^\ell \right) \right\|_{\mathbb{R}^d} \leq \sum_{\ell=1}^m \alpha_\ell \left\| \arg(w^\ell) - \theta^\ell \right\|_{\mathbb{R}^d},$$

and equation (8),

$$d^{\mathbb{T}^d}(z, w) \leq \gamma \sum_{\ell=1}^m \alpha_\ell d_H^{\mathbb{T}^d}(E, F) = \gamma d_H^{\mathbb{T}^d}(E, F).$$

Hence $d^{\mathbb{T}^d}(z, \text{Convex Hull}_{\mathbb{T}^d}(E)) \leq \gamma d_H^{\mathbb{T}^d}(E, F)$.

Exchanging roles of E, F , we have that for every $w \in \text{Convex Hull}_{\mathbb{T}^d}(E)$, the inequality $d^{\mathbb{T}^d}(w, \text{Convex Hull}_{\mathbb{T}^d}(F)) \leq \gamma d_H^{\mathbb{T}^d}(E, F)$ holds. Thus, from the following characterization of the Hausdorff distance (see for instance [5, Exercise 7.3.2]):

(9)

$$d_H^{\mathbb{T}^d}(A, B) \leq r \Leftrightarrow \left(\forall a \in A, d^{\mathbb{T}^d}(a, B) \leq r \text{ and } \forall b \in B, d^{\mathbb{T}^d}(b, A) \leq r \right),$$

we deduce that $d_H^{\mathbb{T}^d}(\text{Convex Hull}_{\mathbb{T}^d}(E), \text{Convex Hull}_{\mathbb{T}^d}(F)) \leq \gamma d_H^{\mathbb{T}^d}(E, F)$. Since $\gamma > 1$ was arbitrary, this concludes the proof. \square

Proof of Lemma 5.15. Since $(E_n)_{n \geq 1}$ converges, it is a Cauchy sequence, and furthermore, thanks to the previous lemma, we have that for every $n \geq 1$, $d_H^{\mathbb{T}^d}(C_n, C_{n+1}) \leq d_H^{\mathbb{T}^d}(E_n, E_{n+1})$. Therefore, the triangle inequality yields, for every $n, p \geq 1$, $d_H^{\mathbb{T}^d}(C_n, C_{n+p}) \leq \sum_{\ell=n}^{n+p-1} d_H^{\mathbb{T}^d}(E_\ell, E_{\ell+1})$. But the series $\sum_{\ell \geq 1} d_H^{\mathbb{T}^d}(E_\ell, E_{\ell+1})$ converges; this implies that the sequence $(C_n)_{n \geq 1}$ is a Cauchy sequence as well. But it is a well-known fact that the set of compact subsets of a complete metric space, endowed with the Hausdorff distance, is complete [5, Proposition 7.3.7]. Applying this to our context, we get that the sequence $(C_n)_{n \geq 1}$ converges to some compact subset $C \subset \mathbb{T}^d$. \square

The limit obtained in Lemma 5.15 is in fact unique, in the sense that it only depends on E and not on the very simple sets used to approximate it.

Lemma 5.17. *If E is a compact connected subset of \mathbb{T}^d and $(E_n)_{n \geq 1}$, $(F_n)_{n \geq 1}$ are two very simple approximations of E (see Lemma 5.15 for terminology), then $\lim_{n \rightarrow +\infty} \text{Convex Hull}_{\mathbb{T}^d}(E_n) = \lim_{n \rightarrow +\infty} \text{Convex Hull}_{\mathbb{T}^d}(F_n)$, where, as usual, the limit is taken with respect to the Hausdorff distance.*

Proof. For $n \geq 1$, let $C_n = \text{Convex Hull}_{\mathbb{T}^d}(E_n)$ and $D_n = \text{Convex Hull}_{\mathbb{T}^d}(F_n)$. Denote by C (respectively D) the limit of the sequence $(C_n)_{n \geq 1}$ (respectively $(D_n)_{n \geq 1}$). By the triangle inequality, we have that

$$(10) \quad d_H^{\mathbb{T}^d}(C, D) \leq d_H^{\mathbb{T}^d}(C, C_n) + d_H^{\mathbb{T}^d}(C_n, D_n) + d_H^{\mathbb{T}^d}(D_n, D).$$

The first and third terms on the right hand side of this inequality go to zero as n goes to infinity. Let us estimate the second one. For every $p \geq n$, we have, using the triangle inequality again:

$$\begin{aligned} d_H^{\mathbb{T}^d}(E_n, F_n) &\leq \frac{1}{2^n} \left(\min \left(1, d \left(\arg(E_n), \partial \left([-\pi, \pi]^d \right) \right) \right) \right. \\ &\quad \left. + \min \left(1, d \left(\arg(F_n), \partial \left([-\pi, \pi]^d \right) \right) \right) \right) + d_H^{\mathbb{T}^d}(E_{n+1}, F_{n+1}). \end{aligned}$$

By iterating and using the fact that

$$d\left(\arg(E_{n+1}), \partial\left([- \pi, \pi]^d\right)\right) \leq d\left(\arg(E_n), \partial\left([- \pi, \pi]^d\right)\right)$$

since $E_n \subset E_{n+1}$, we obtain that for every $p \geq 1$

$$\begin{aligned} d_H^{\mathbb{T}^d}(E_n, F_n) &\leq \frac{1}{2^n} \left(\sum_{\ell=0}^p \frac{1}{2^\ell} \right) \left(\min\left(1, d\left(\arg(E_n), \partial\left([- \pi, \pi]^d\right)\right)\right) \right. \\ &\quad \left. + \min\left(1, d\left(\arg(F_n), \partial\left([- \pi, \pi]^d\right)\right)\right) \right) + d_H^{\mathbb{T}^d}(E_{n+p}, F_{n+p}). \end{aligned}$$

Letting p go to infinity, we deduce that

$$d_H^{\mathbb{T}^d}(E_n, F_n) \leq \frac{1}{2^{n-1}} \max\left(d\left(\arg(E_n), \partial\left([- \pi, \pi]^d\right)\right), d\left(\arg(F_n), \partial\left([- \pi, \pi]^d\right)\right)\right).$$

Therefore, thanks to Lemma 5.16, the second term in the right hand side of equation (10) satisfies $d_H^{\mathbb{T}^d}(C_n, D_n) \leq d_H^{\mathbb{T}^d}(E_n, F_n)$ and converges to zero as well. Hence, letting n go to infinity in equation (10), we obtain that $d_H^{\mathbb{T}^d}(C, D) = 0$, which yields $C = D$ because C and D are closed. \square

Now, let E be any compact connected subset of \mathbb{T}^d , and let $\text{App}(E) \subset \mathbb{T}^d$ be the set of $a \in \mathbb{T}^d$ such that $a \cdot E$ admits a very simple approximation (see Lemma 5.15). We define a map C_E from $\text{App}(E)$ to the set of compact subsets of \mathbb{T}^d as follows: for $a \in \text{App}(E)$, $C_E(a)$ is the limit of the convex hull of any very simple approximation of $a \cdot E$.

Definition 5.18. We say that E is *hullizable* if $\text{App}(E) \neq \emptyset$ and the map $a \in \text{App}(E) \mapsto a^{-1} \cdot C_E(a)$ is constant.

Figure 4 displays an example of non-hullizable set.

Definition 5.19. Let E be a hullizable compact connected subset of \mathbb{T}^d . We define the convex hull of E as $\text{Convex Hull}_{\mathbb{T}^d}(E) := a^{-1} \cdot C_E(a)$ for any $a \in \text{App}(E)$.

This definition agrees with Definition 5.7 when E is simple. Indeed, if $a \in \mathbb{T}^d$ is such that $a \cdot E$ is very simple, then $(F_n = a \cdot E)_{n \geq 1}$ is a very simple approximation of $a \cdot E$, hence

$$C_E(a) = \text{Convex Hull}_{\mathbb{T}^d}(a \cdot E) = \exp(i \text{Convex Hull}(\arg(a \cdot E)))$$

Consequently, it follows from Lemma 5.9 (with $N = 1$) that E is hullizable, and its convex hull is computed as in Definition 5.7.

6. PROOF OF THE MAIN THEOREM

In this section, we prove our main result (Theorem 6.6). In the proof we use the results proved in [21] for the self-adjoint case.

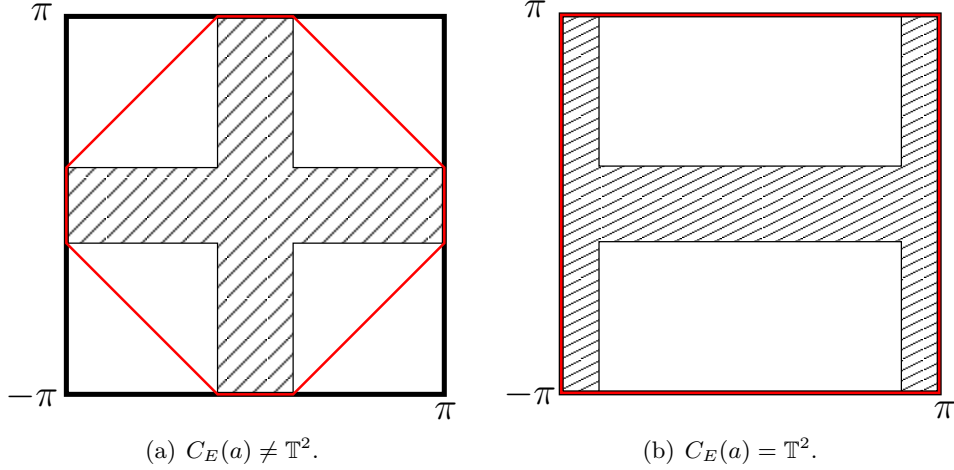


FIGURE 4. An example of non hullizable set $E \subset \mathbb{T}^2$. The figure displays $\arg(a \cdot E)$ for two different values of $a \in \mathbb{T}^2$, and the corresponding set $\arg(C_E(a))$ (the boundary of which is represented by a red line).

6.1. The inverse Cayley transform for unitary operators. Let us recall the definition of the inverse Cayley transform of a unitary operator [24, Definition 3.17]. Let U be a unitary operator such that $-1 \notin \text{Sp}(U)$. We define the inverse Cayley transform of U as $\mathcal{C}(U) = i(\text{Id} - U)(\text{Id} + U)^{-1}$. Then $\mathcal{C}(U)$ is a self-adjoint operator.

Lemma 6.1. *Let U, V be commuting unitary operators acting on a Hilbert space \mathcal{H} , none of them having -1 in its spectrum. Then $\mathcal{C}(U)$ and $\mathcal{C}(V)$ commute.*

Proof. This is a consequence of the following fact: if A is a normal operator acting on a Hilbert space \mathcal{H} , with spectral measure E_A , S is a Borel set and $f : \mathbb{C} \rightarrow \mathbb{C}$ is a measurable function, then $E_{f(A)}(S) = E_A(f^{-1}(S))$. Therefore, if B is another normal operator which commutes with A and g is another measurable function, the spectral projections $E_{f(A)}(S)$ and $E_{g(B)}(T)$ commute for every Borel sets S, T . Hence $f(A)$ and $g(B)$ commute. \square

Consequently, if U_1, \dots, U_d are commuting unitary operators, it makes sense to talk about the joint spectrum of the family $\mathcal{C}(U_1), \dots, \mathcal{C}(U_d)$. We recall that the joint spectrum of a finite family of pairwise commuting normal operators is defined as the support of its joint spectral measure.

Lemma 6.2. *Let U_1, \dots, U_d be commuting unitary operators acting on a Hilbert space \mathcal{H} , none of them having -1 in its spectrum. Then*

$$\text{JointSpec}(\mathcal{C}(U_1), \dots, \mathcal{C}(U_d)) = \overline{\left\{ \frac{1}{2} \arg(\lambda), \lambda \in \text{JointSpec}(U_1, \dots, U_d) \right\}}.$$

Proof. We mimic the reasoning of the proof of Proposition 5.25 in [26] (which deals with the spectrum of one single operator). For every $j \in \llbracket 1, d \rrbracket$, we have that $\mathcal{C}(U_j) = \phi(U_j)$ with

$$\phi : \mathbb{C} \setminus \{-1\} \rightarrow \mathbb{C}, \quad z \mapsto i \frac{1-z}{1+z}.$$

Let $\mu = E_{U_1} \otimes \dots \otimes E_{U_d}$ be the joint spectral measure of U_1, \dots, U_d , and let $\nu = E_{\mathcal{C}(U_1)} \otimes \dots \otimes E_{\mathcal{C}(U_d)}$ be the joint spectral measure of $\mathcal{C}(U_1), \dots, \mathcal{C}(U_d)$; we need to prove that $\text{supp}(\nu) = \overline{\{(\phi(\lambda_1), \dots, \phi(\lambda_d)), \lambda \in \text{supp}(\mu)\}} =: S$. Indeed, a straightforward computation shows that for every $z \in \mathbb{S}^1 \setminus \{-1\}$, $\phi(z) = \frac{1}{2} \arg z$. Firstly, let $\zeta = (\zeta_1, \dots, \zeta_d) \in S$, and let $\varepsilon_1, \dots, \varepsilon_d > 0$ be small enough; there exists $\lambda = (\lambda_1, \dots, \lambda_d) \in \text{supp}(\mu)$ such that for every $j \in \llbracket 1, d \rrbracket$, the inequality $|\zeta_j - \lambda_j| < \varepsilon_j$ holds. Since ϕ is continuous in a neighborhood of $\text{Sp}(U_j)$ in \mathbb{S}^1 (because $\text{Sp}(U_j)$ is closed and does not contain -1), there exists $\delta_j > 0$ such that

$$D(\lambda_j, \delta_j) \subset \{z \in \mathbb{C}, |\phi(z) - \phi(\lambda_j)| < \varepsilon_j\} \subset \phi^{-1}(D(\zeta_j, 2\varepsilon_j))$$

where $D(z, r)$ stands for the open disk of radius r centered at z . We deduce from this inclusion that $E_{U_j}(\phi^{-1}(D(\zeta_j, 2\varepsilon_j))) \geq E_{U_j}(D(\lambda_j, \delta_j)) > 0$, where the last inequality comes from the fact that λ belongs to the support of E_{U_j} . Consequently, if $D := D(\zeta_1, 2\varepsilon_1) \times \dots \times D(\zeta_d, 2\varepsilon_d)$, we have that

$$\nu(D) = \prod_{j=1}^d E_{\phi(U_j)}(D(\zeta_j, 2\varepsilon_j)) = \prod_{j=1}^d E_{U_j}(\phi^{-1}(D(\zeta_j, 2\varepsilon_j))) > 0,$$

which means that ζ belongs to the support of ν .

Conversely, if $\zeta \notin S$, there exists $j \in \llbracket 1, d \rrbracket$ such that $\phi^{-1}(D(\zeta_j, \varepsilon_j))$ is empty for every $\varepsilon_j > 0$ small enough, and we conclude with similar computations that $\zeta \notin \text{supp}(\nu)$. □

6.2. A refinement of Theorem 2.3. We can make an improvement on Theorem 2.3 that we will use in the proof of our main result.

Theorem 6.3. *Let $A_1(\hbar), \dots, A_d(\hbar)$ be pairwise commuting self-adjoint operators acting on \mathcal{H}_\hbar , and let $T_1(\hbar), \dots, T_d(\hbar)$ be self-adjoint semiclassical operators acting on \mathcal{H}_\hbar , with bounded principal symbols f_1, \dots, f_d . Assume moreover that for all $j \in \llbracket 1, d \rrbracket$, $\|T_j(\hbar) - A_j(\hbar)\| = \mathcal{O}(\hbar)$. Then*

$$\text{Convex Hull}(\text{JointSpec}(A_1(\hbar), \dots, A_d(\hbar))) \xrightarrow{\hbar \rightarrow 0} \overline{\text{Convex Hull}(\mathcal{F}(\mathcal{M}))}$$

where $F = (f_1, \dots, f_d) : M \rightarrow \mathbb{R}^d$.

Since the proof is close to the one of the aforementioned theorem, we will assume some degree of familiarity with the content of [21]. The first step is to prove the following result comparing only two operators.

Lemma 6.4. *Let A_h, T_h be self-adjoint operators acting on \mathcal{H}_h such that T_h is a semiclassical operator with principal symbol f_0 and $\|A_h - T_h\| = \mathcal{O}(\hbar)$. Let $\lambda_{\sup}(\hbar) = \sup \operatorname{Sp}(A_h)$, which may be infinite. Then $\lambda_{\sup}(\hbar) \xrightarrow{\hbar \rightarrow 0} \sup_M f_0$.*

Proof. Let $R_h = A_h - T_h$, so that $\|R_h\| = \mathcal{O}(\hbar)$ by assumption; we choose $\hbar_0 \in I$ and $C > 0$ such that $\|R_h\| \leq C\hbar$ for every $\hbar \leq \hbar_0$. We also introduce the (possibly infinite) quantity $\mu_{\sup}(\hbar) = \sup \operatorname{Sp}(T_h)$. Our goal is to compare $\lambda_{\sup}(\hbar)$ to $\mu_{\sup}(\hbar)$; of course, thanks to Equation (1), we have that

$$\lambda_{\sup}(\hbar) = \sup_{v \in \mathcal{H}_h, \|v\|=1} \langle A_h v, v \rangle, \quad \mu_{\sup}(\hbar) = \sup_{v \in \mathcal{H}_h, \|v\|=1} \langle T_h v, v \rangle.$$

Let $\hbar \leq \hbar_0$, and let us start with the case where $\mu_{\sup}(\hbar) = +\infty$. Let $M > 0$; there exists $v_0 \in \mathcal{H}_h$ with unit norm such that $\langle T_h v_0, v_0 \rangle \geq M$; this yields

$$\langle A_h v_0, v_0 \rangle = \langle T_h v_0, v_0 \rangle + \langle R_h v_0, v_0 \rangle \geq M - C\hbar.$$

Since M is arbitrarily large, this means that $\lambda_{\sup}(\hbar) = +\infty$. Now, we assume that $\mu_{\sup}(\hbar)$ is finite. From the equality

$$\lambda_{\sup}(\hbar) = \sup_{v \in \mathcal{H}_h, \|v\|=1} (\langle T_h v, v \rangle + \langle R_h v, v \rangle),$$

we derive that

$$\lambda_{\sup}(\hbar) \leq \mu_{\sup}(\hbar) + \sup_{v \in \mathcal{H}_h, \|v\|=1} \langle R_h v, v \rangle \leq \mu_{\sup}(\hbar) + C\hbar.$$

Moreover, there exists a unit vector $v_0 \in \mathcal{H}$ such that $\mu_{\sup}(\hbar) \leq \langle T_h v_0, v_0 \rangle + \hbar$. By decomposing

$$\langle A_h v_0, v_0 \rangle = \langle R_h v_0, v_0 \rangle + \langle T_h v_0, v_0 \rangle - \mu_{\sup}(\hbar) + \mu_{\sup}(\hbar),$$

we get that $\lambda_{\sup}(\hbar) \geq \langle A_h v_0, v_0 \rangle \geq \mu_{\sup}(\hbar) - (C+1)\hbar$, so finally

$$\mu_{\sup}(\hbar) - (C+1)\hbar \leq \lambda_{\sup}(\hbar) \leq \mu_{\sup}(\hbar) + C\hbar.$$

Therefore, the result comes from the fact that $\mu_{\sup}(\hbar)$ tends to $\sup_M f_0$ as \hbar goes to zero [21, Lemma 11]. \square

Proof of Theorem 6.3. We follow the reasoning of the proof of [21, Theorem 8]. More precisely, let $\Sigma_h = \operatorname{JointSpec}(A_1(\hbar), \dots, A_d(\hbar))$ and consider, for any subset S of \mathbb{R}^d , the function

$$\Phi_S : \mathbb{S}^{d-1} \rightarrow \mathbb{R} \cup \{+\infty\}, \quad \alpha \mapsto \sup_{x \in S} \sum_{j=1}^d \alpha_j x_j.$$

Then it suffices to show that Φ_{Σ_h} converges uniformly to $\Phi_{F(M)}$ as \hbar goes to zero. We start by proving the pointwise convergence. Let $\alpha \in \mathbb{S}^{d-1}$ and consider the self-adjoint operator $A_h^{(\alpha)} = \sum_{j=1}^d \alpha_j A_j(\hbar)$; by [21, Lemma 14], $\Phi_{\Sigma_h}(\alpha) = \sup \operatorname{Sp}(A_h^{(\alpha)})$. In a similar fashion, we introduce the operator

$T_h^{(\alpha)} = \sum_{j=1}^d \alpha_j T_j(\hbar)$ and the function $f^{(\alpha)} = \sum_{j=1}^d \alpha_j f_j$, so that $T_h^{(\alpha)}$ is a self-adjoint semiclassical operator with principal symbol $f^{(\alpha)}$. Furthermore, since $\|T_j(\hbar) - A_j(\hbar)\| = \mathcal{O}(\hbar)$ for $j = 1, \dots, d$, we also have the estimate $\|T^{(\alpha)}(\hbar) - A^{(\alpha)}(\hbar)\| = \mathcal{O}(\hbar)$. Consequently, it follows from the previous lemma that

$$\Phi_{\Sigma_h}(\alpha) = \sup \operatorname{Sp} \left(A_h^{(\alpha)} \right) \xrightarrow{\hbar \rightarrow 0} \sup_M f^{(\alpha)} = \Phi_{F(M)}(\alpha).$$

To prove that this convergence is uniform, we observe that the boundedness of the principal symbols f_1, \dots, f_d implies the boundedness of $T_1(\hbar), \dots, T_d(\hbar)$, which in turn implies the boundedness of $A_1(\hbar), \dots, A_d(\hbar)$. Therefore the joint spectrum of the latter family is bounded, hence compact. We conclude by the argument used in the last part of the proof of Theorem 8 in [21]. \square

6.3. When none of the principal symbols is onto. In this section, we consider pairwise commuting unitary semiclassical operators $U_1(\hbar), \dots, U_d(\hbar)$ with joint principal symbol $F = (f_0^1, \dots, f_0^d)$. We assume that for every $j \in \llbracket 1, d \rrbracket$, $f_0^j(M)$ is closed, and that the same holds for $F(M)$. We assume moreover that none of the principal symbols $f_0^j : M \rightarrow \mathbb{S}^1$, $j \in \llbracket 1, d \rrbracket$, is onto; using the terminology introduced earlier, this means that $F(M)$ is a simple compact subset of \mathbb{T}^d . Note that this set is connected since it is the image of M , which is itself connected, by a continuous function.

Let us introduce an additional assumption in the case where the joint spectrum of $(U_1(\hbar), \dots, U_d(\hbar))$ is generic (see Lemma 5.11):

- (A1) There is $\hbar_0 \in I$ and a point $b \in \mathbb{T}^d$ which is admissible (see Lemma 5.6 for the terminology) for all $\operatorname{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))$, $\hbar \leq \hbar_0$, and such that $b \cdot F(M)$ is very simple.

Remark 6.5. This assumption might seem strange but will be crucial for a part of our analysis. Indeed, it may not hold if the joint spectrum is too sparse (see Figure 5). In this situation, given the data of the joint spectrum only, its convex hull computed thanks to our definition will be far from the convex hull of $F(M)$. However, this assumption is reasonable, because it holds for Berezin-Toeplitz and pseudodifferential operators, as a corollary of the Bohr-Sommerfeld rules which imply that the joint spectrum is “dense” (when $\hbar \rightarrow 0$) in the set of regular values of F (see [14] for pseudodifferential operators and [7] for Berezin-Toeplitz operators). Nevertheless, our assumption is much weaker than the Bohr-Sommerfeld rules. \oslash

Now, we do not necessarily assume that the joint spectrum is generic anymore. Our goal is to prove the following result.

Theorem 6.6. *For every $b \in \mathbb{T}^d$ such that $b \cdot F(M)$ is very simple,*

$$b^{-1} \cdot \exp(i \operatorname{Convex Hull}(\arg(b \cdot \operatorname{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))))))$$

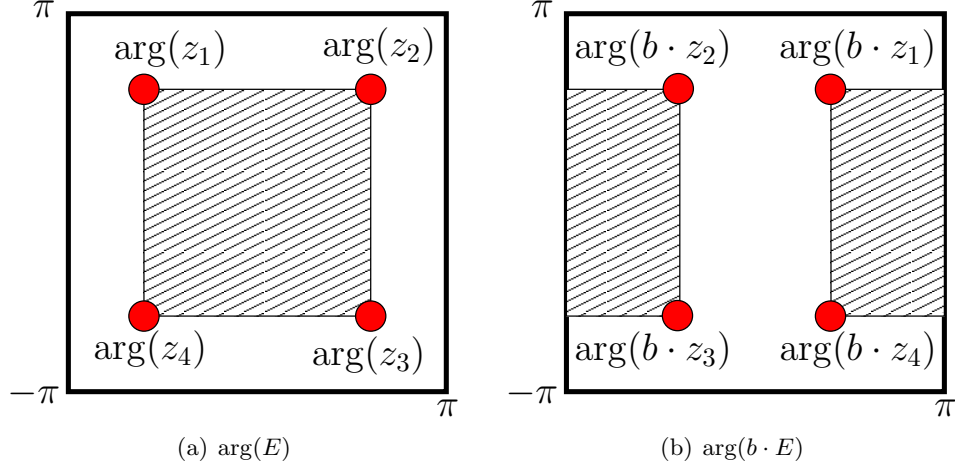


FIGURE 5. An example for which assumption (A1) does not hold.

converges, when $\hbar \rightarrow 0$, to $\overline{\text{Convex Hull}_{\mathbb{T}^d}(F(M))}$ with respect to the Hausdorff distance on \mathbb{T}^d . In particular, if the joint spectrum is generic and assumption (A1) holds, then

$$(11) \quad \text{Convex Hull}_{\mathbb{T}^d}(\text{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))) \xrightarrow{\hbar \rightarrow 0} \overline{\text{Convex Hull}_{\mathbb{T}^d}(F(M))}.$$

In this statement, we use that $b \cdot \text{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))$ is very simple, for $\hbar \in I$ small enough, whenever $b \cdot F(M)$ is very simple. This is a consequence of the following lemma.

Lemma 6.7. *Let j in $\llbracket 1, d \rrbracket$, and let $a \in \mathbb{S}^1 \setminus f_0^j(M)$. Then there exists $\hbar_0 \in I$ such that for every $\hbar \leq \hbar_0$ in I , $a \notin \text{Sp}(U_j(\hbar))$.*

Proof. This is a consequence of Corollary 4.4 (more precisely, of its consequence stated right after the proof of Lemma 4.8). Indeed, since $f_0^j(M)$ is closed, there exists a small open neighborhood of a in \mathbb{S}^1 not intersecting it. Thus there exists $c > 0$ such that $|f_0^j - a| \geq c$. Hence the operator $U_j(\hbar) - a \text{Id}$ is invertible, so a does not belong to the spectrum of $U_j(\hbar)$. \square

Lemma 6.8. *Let $(U_\hbar)_{\hbar \in I}$ be a unitary semiclassical operator with principal symbol $f_0 : M \rightarrow \mathbb{S}^1$ such that $f_0(M)$ is closed and does not contain -1 . Then $\|\mathcal{C}(U_\hbar) - \text{Op}_\hbar(\phi \circ f_0)\| = \mathcal{O}(\hbar)$, where $\phi : \mathbb{C} \setminus \{-1\} \rightarrow \mathbb{C}$ is $z \mapsto i \frac{1-z}{1+z}$.*

Note that this statement makes sense since by the above lemma, there exists $\hbar_0 \in I$ such that for every $\hbar \leq \hbar_0$, $-1 \notin \text{Sp}(U_\hbar)$.

Proof. By the same argument that we have used in the proof of the previous lemma, $\text{Op}_\hbar(1 + f_0)$ is invertible and the norm of its inverse is uniformly bounded in \hbar . Thus by axiom (Q1) and Remark 4.5, we have that

$$\|\text{Op}_\hbar(\phi \circ f_0) - i \text{Op}_\hbar(1 - f_0) \text{Op}_\hbar(1 + f_0)^{-1}\| = \mathcal{O}(\hbar),$$

which yields by axiom **(Q3)**:

$$(12) \quad \left\| \text{Op}_h(\phi \circ f_0) - i (\text{Id} - \text{Op}_h(f_0))(\text{Id} + \text{Op}_h(f_0))^{-1} \right\| = \mathcal{O}(\hbar).$$

Furhermore, $\text{Id} + U_h = \text{Id} + \text{Op}_h(f_0) + R_h$ with $\|R_h\| = \mathcal{O}(\hbar)$. Consequently (see e.g. [15, Theorem A3.31]), $(\text{Id} + U_h)^{-1} = (\text{Id} + \text{Op}_h(f_0))^{-1} (\text{Id} + A_h)^{-1}$ where $A_h = R_h (\text{Id} + \text{Op}_h(f_0))^{-1}$; observe that $\|A_h\| = \mathcal{O}(\hbar)$. We derive from the above equation the inequality

$$\left\| (\text{Id} + U_h)^{-1} - (\text{Id} + \text{Op}_h(f_0))^{-1} \right\| \leq \left\| (\text{Id} + \text{Op}_h(f_0))^{-1} \right\| \left\| (\text{Id} + A_h)^{-1} - \text{Id} \right\|.$$

But we have that

$$\left\| (\text{Id} + A_h)^{-1} - \text{Id} \right\| \leq \sum_{n=1}^{+\infty} \|A_h\|^n = \frac{1}{1 - \|A_h\|} - 1 = \mathcal{O}(\hbar).$$

Therefore we finally obtain that $\left\| (\text{Id} + U_h)^{-1} - (\text{Id} + \text{Op}_h(f_0))^{-1} \right\| = \mathcal{O}(\hbar)$. Since obviously $\left\| (\text{Id} - U_h) - (\text{Id} - \text{Op}_h(f_0)) \right\| = \mathcal{O}(\hbar)$, Equation (12) and the triangle inequality finally yield

$$\left\| \text{Op}_h(\phi \circ f_0) - i (\text{Id} - U_h)(\text{Id} + U_h)^{-1} \right\| = \mathcal{O}(\hbar),$$

which was to be proved. \square

Before proving Theorem 6.6, we state one last technical lemma.

Lemma 6.9. *Let E be a compact subset of $(-\pi, \pi)^d$ and let $(E_\varepsilon)_{\varepsilon>0}$ be a family of compact subsets of $(-\pi, \pi)^d$ such that $d_H(E, E_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$. Then $d_H^{\mathbb{T}^d}(\exp(iE), \exp(iE_\varepsilon)) \xrightarrow{\varepsilon \rightarrow 0} 0$.*

Proof. Let $\delta_0 = d(E, \partial([-\pi, \pi]^d))$ be the distance between E and the boundary of $[-\pi, \pi]^d$ in \mathbb{R}^d . Choose a positive number $\delta \leq \frac{1}{2}\delta_0$; there exists $\varepsilon > 0$ such that $d_H(E, E_\varepsilon) \leq \delta$. Let γ be such that $1 < \gamma < 2$. Let $u \in E$; by definition of the Hausdorff distance, there exists $v \in E_\varepsilon$ such that $\|u - v\|_{\mathbb{R}^d} \leq \gamma d_H(E, E_\varepsilon)$. Now, let $\theta \in (2\pi\mathbb{Z})^d$ be non-zero; then $v - \theta$ does not belong to $[-\pi, \pi]^d$, thus $\|u - v + \theta\|_{\mathbb{R}^d} \geq \delta_0 \geq \gamma\delta \geq \|u - v\|_{\mathbb{R}^d}$. Consequently, we have that

$$d^{\mathbb{T}^d}(\exp(iu), \exp(iv)) = \|u - v\|_{\mathbb{R}^d} \leq \gamma d_H(E, E_\varepsilon) \leq \gamma\delta.$$

Therefore $d^{\mathbb{T}^d}(\exp(iu), \exp(iE_\varepsilon)) \leq \gamma\delta$. Exchanging the roles of E and E_ε , we also get that for every v in E_ε , $d^{\mathbb{T}^d}(\exp(iv), \exp(iE)) \leq \gamma\delta$. This implies that $d_H^{\mathbb{T}^d}(\exp(iE), \exp(iE_\varepsilon)) \leq \gamma\delta$, because of the characterization (9). \square

We are finally ready to give a proof of the main result of this section.

Proof of Theorem 6.6. Let $b = (b_1, \dots, b_d) \in \mathbb{T}^d$ be such that $b \cdot F(M)$ is very simple. For every $j \in \llbracket 1, d \rrbracket$, we consider the operator $V_j(\hbar) = b_j U_j(\hbar)$, which is a semiclassical unitary operator, with principal symbol $g_0^j = b_j f_0^j$. By Lemma 6.7, there exists $\hbar_j \in I$ such that $-1 \notin \text{Sp}(V_j(\hbar))$ whenever

$\hbar \leq \hbar_j$. Let $\hbar_0 = \min_{1 \leq j \leq d} \hbar_j$; in the rest of the proof we will assume that $\hbar \leq \hbar_0$. We can therefore consider the self-adjoint operators

$$T_j(\hbar) = 2 \mathcal{C}(V_j(\hbar)), \quad 1 \leq j \leq d,$$

where we recall that \mathcal{C} stands for the inverse Cayley transform (see Section 6.1). By Lemma 6.1, $T_j(\hbar)$ and $T_m(\hbar)$ commute for every $j, m \in \llbracket 1, d \rrbracket$. We also consider the self-adjoint semiclassical operators $B_j(\hbar) = \text{Op}_\hbar(a_0^j)$, $1 \leq j \leq d$, where $a_0^j = 2\phi \circ g_0^j$. We also recall that for $z \in \mathbb{S}^1 \setminus \{-1\}$, $\phi(z) = \frac{1}{2} \arg z$, and thus $a_0^j = \arg g_0^j = \arg(b_j f_0^j)$. Let $A = (a_0^1, \dots, a_0^d)$. Since by Lemma 6.8, $\|T_j(\hbar) - B_j(\hbar)\| = \mathcal{O}(\hbar)$, for every $j \in \llbracket 1, d \rrbracket$, Theorem 6.3 implies that

$$(13) \quad \text{Convex Hull}(\text{JointSpec}(T_1(\hbar), \dots, T_d(\hbar))) \xrightarrow{\hbar \rightarrow 0} \overline{\text{Convex Hull}(A(M))}$$

with respect to the Hausdorff distance on \mathbb{R}^d . On the one hand, we have that $\text{Convex Hull}(A(M)) = \text{Convex Hull}(\arg(b \cdot F(M)))$. On the other hand, Lemma 6.2 yields

$$\text{JointSpec}(T_1(\hbar), \dots, T_d(\hbar)) = \overline{\arg(\text{JointSpec}(V_1(\hbar), \dots, V_d(\hbar)))}.$$

Substituting these results in equation (13), we obtain that

$$\text{Convex Hull}(b \cdot \arg(\text{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))))$$

converges, when \hbar goes to zero, to $\overline{\text{Convex Hull}(b \cdot \arg(F(M)))}$ with respect to the Hausdorff distance on \mathbb{R}^d . By Lemma 6.9, this in turn implies that $\exp(i \text{Convex Hull}(b \cdot \arg(\text{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))))$ converges to $\exp(i \overline{\text{Convex Hull}(b \cdot \arg(F(M)))})$ for the Hausdorff distance on \mathbb{T}^d when \hbar goes to zero. Using the continuity of \exp and of the restriction of \arg to $(-\pi, \pi)^d$, we see that the latter is $\exp(i \text{Convex Hull}(b \cdot \arg(F(M))))$. Finally, using that $z \in \mathbb{T}^d \mapsto b^{-1} \cdot z$ is continuous and preserves the Hausdorff distance (Lemma 5.1), this yields the first part of the Theorem.

For the second statement of the Theorem, we apply the first part with a point $b = (b_1, \dots, b_d) \in \mathbb{T}^d$ which is admissible for all the joint spectra $\text{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))$, $\hbar \leq \hbar_0$, and such that the set $b \cdot F(M)$ is very simple, keeping in mind Definition 5.7. \square

6.4. A conjecture in the general case. We would like to get rid of the assumption on the surjectivity of the principal symbols. We consider pairwise commuting unitary semiclassical operators $U_1(\hbar), \dots, U_d(\hbar)$ with joint principal symbol $F = (f_0^1, \dots, f_0^d)$. We still assume that $F(M)$ is closed.

Conjecture 6.10. *Assume that $F(M)$ is hullizable (see Definition 5.18). From the behaviour of the joint spectrum $\text{JointSpec}(U_1(\hbar), \dots, U_d(\hbar))$ when \hbar goes to zero, one can recover the convex hull of $F(M)$.*

We give evidence for this conjecture in Section 7.3, but first let us make a few comments about it. Firstly, “recover” can have several meanings, but it would be appreciable to obtain a statement similar to Theorem 6.6 involving

the convex hull of the joint spectrum; however, the latter may no longer be simple, so we would need to give a meaning to its convex hull. Secondly, in order to prove this conjecture, using axioms (Q1) to (Q6) only might not be enough, thus a natural problem would be to look for the minimal set of additional axioms needed for this proof.

7. A SEMICLASSICAL APPROACH TO NON HAMILTONIAN SYMPLECTIC ACTIONS: QUANTIZATION OF CIRCLE-VALUED MOMENTUM MAPS

Symplectic actions that are not Hamiltonian have recently become of important relevance in view of the work of Susan Tolman [28] (which constructs many such actions with isolated fixed points on compact manifolds). In this case there is no momentum map in the usual sense, but one can construct a circle-valued function playing the same part.

7.1. Construction of the circle valued momentum map. We identify \mathbb{S}^1 with \mathbb{R}/\mathbb{Z} and denote by $\pi : \mathbb{R} \ni t \mapsto [t] \in \mathbb{R}/\mathbb{Z}$ the projection. The length form $\lambda \in \Omega^1(\mathbb{R}/\mathbb{Z})$ is given by $\lambda([t])(T_t\pi(r)) := r$. Let (M, ω) be a connected symplectic manifold, that is, M is a smooth manifold and ω is a smooth 2-form on M which is non-degenerate and closed. Let $\Phi : (\mathbb{R}/\mathbb{Z}) \times M \rightarrow M$ be a smooth symplectic action, that is a smooth action by diffeomorphisms $\Phi_{[t]} : M \rightarrow M$ that preserves the symplectic form ω (these are called *symplectomorphisms*). For $r \in \mathbb{R}$ denote by $r_M \in \mathfrak{X}(M)$ the action infinitesimal generator given by $r_M(x) := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \Phi_{[r\varepsilon]}(x)$.

Definition 7.1. The \mathbb{R}/\mathbb{Z} -action on (M, ω) is *Hamiltonian* if there is a smooth map $\mu : M \rightarrow \mathbb{R}$ such that $\mathbf{i}_{1_M}\omega := \omega(1_M, \cdot) = d\mu$. The map μ is called the *momentum map* of the action.

Note that the existence of μ is equivalent to the one-form $\mathbf{i}_{1_M}\omega$ being exact, and therefore if the first cohomology group $H^1(M; \mathbb{R})$ vanishes then every symplectic \mathbb{R}/\mathbb{Z} -action on M is in fact Hamiltonian.

If the \mathbb{R}/\mathbb{Z} -action does not have a momentum map in the sense above, then the action must be non trivial. Hence, if the action is not Hamiltonian, then $\mathbf{i}_{1_M}\omega$ is not exact. These type of actions also admit an analogue of the momentum map, called the *circle valued momentum map*, and which now takes values in \mathbb{R}/\mathbb{Z} . A *circle valued momentum map* $\mu : M \rightarrow \mathbb{R}/\mathbb{Z}$ is determined by the equation $\mu^*\lambda = \mathbf{i}_{1_M}\omega$.

Such a map μ always exists, for either ω itself, or a very close perturbation of it. To be more precise, suppose that \mathbb{R}/\mathbb{Z} acts symplectically on the closed symplectic manifold (M, ω) , but not Hamiltonianly. Whenever the symplectic form ω is integral (that is, $[\omega] \in H^2(M; \mathbb{Z})$), then the action admits a circle valued momentum map $\mu : M \rightarrow \mathbb{R}/\mathbb{Z}$ for ω (this result is due to McDuff, see [19], and is valid for some symplectic form even when the integral cohomology assumption is invalid).

For the sake of completeness and because it is a very simple construction, we review it here. It follows from [22, Lemma 7] that $[\mathbf{i}_{1_M}\omega] \in H^1(M; \mathbb{Z})$.

Fix $m_0 \in M$ and let γ_m be an arbitrary smooth path in M , from m_0 to m , and define $\mu : M \rightarrow \mathbb{R}/\mathbb{Z}$ by

$$(14) \quad \mu(m) := \left[\int_{\gamma_m} \mathbf{i}_{1_M} \omega \right].$$

It is immediate that the definition of μ is independent of paths, so it is well defined. Also, μ is clearly smooth, and for every $v_m \in T_m M$, we have $T_m \mu(v_m) = T_{\int_{\gamma_m} \mathbf{i}_{1_M} \omega} \pi(\mathbf{i}_{1_M} \omega(m)(v_m))$, and consequently $(\mu^* \lambda)(m)(v_m) = \lambda(\mu(m))(T_m \mu(v_m)) = (\mathbf{i}_{1_M} \omega)(m)(v_m)$, as desired. The map μ is defined up to the addition of constants (due to the freedom in the choice of m_0).

7.2. Circle action and Berezin-Toeplitz quantization. It turns out that there exists a natural way to derive a semiclassical quantization of this circle-valued moment map when M is compact and ω is integral (in fact, integral up to a factor 2π); this semiclassical quantization is called Berezin-Toeplitz quantization. It builds on geometric quantization, due to Kostant [17] and Souriau [27]. Berezin-Toeplitz operators were introduced by Berezin [1], their microlocal analysis was initiated by Boutet de Monvel and Guillemin [4], and they have been studied by many authors since (see for instance the review [25] and the references therein).

Assume that (M, ω) is a compact, connected, Kähler manifold, which means that it is endowed with an almost complex structure which is compatible with ω and integrable. We recall that an almost complex structure j on M is a smooth section of the bundle $\text{End}(TM) \rightarrow M$ such that $j^2 = -\text{id}_{TM}$, and j being integrable means that it induces on M a structure of complex manifold. Compatibility between ω and j means that $\omega(\cdot, j\cdot)$ is a Riemannian metric on M .

Assume that the cohomology class $[\omega/2\pi]$ lies in $H^2(M, \mathbb{Z})$. Then there exists a prequantum line bundle $L \rightarrow M$, that is a holomorphic, Hermitian complex line bundle whose Chern connection (the unique connection compatible with both the holomorphic and Hermitian structures) has curvature form equal to $-i\omega$. Then for any integer $k \geq 1$, the space

$$\mathcal{H}_k = H^0(M, L^{\otimes k})$$

of holomorphic sections of the line bundle $L^{\otimes k} \rightarrow M$, endowed with the Hermitian product

$$\phi, \psi \in \mathcal{H}_k \mapsto \langle \phi, \psi \rangle_k = \int_M h_k(\phi, \psi) \mu_M$$

where μ_M is the Liouville measure associated with ω and h_k is the Hermitian form on $L^{\otimes k}$ inherited from the one of L , is a finite dimensional Hilbert space.

Now, the quantization map $\text{Op}_k : \mathcal{C}^\infty(M, \mathbb{C}) \rightarrow \mathcal{L}(\mathcal{H}_k)$ is defined as follows: let $L^2(M, L^{\otimes k})$ be the space of square integrable sections of the line bundle $L^{\otimes k} \rightarrow M$, that is the completion of $\mathcal{C}^\infty(M, L^{\otimes k})$ with respect to $\langle \cdot, \cdot \rangle_k$, and let Π_k be the orthogonal projector from $L^2(M, L^{\otimes k})$ to \mathcal{H}_k .

Then, given $f \in \mathcal{C}^\infty(M, \mathbb{C})$, let $\text{Op}_k(f) = \Pi_k f$ where, by a slight abuse of notation, f stands for the operator of multiplication by f in $L^2(M, L^{\otimes k})$. Here the integer parameter k plays the part of the inverse of \hbar , therefore the semiclassical limit corresponds to $k \rightarrow +\infty$ instead of $\hbar \rightarrow 0$.

Lemma 7.2. *The Berezin-Toeplitz quantization is a semiclassical quantization in the sense of Section 4.2.*

Proof. This work was done in [21] for axioms (Q3) to (Q6). The fact that axiom (Q1) is satisfied comes, for instance, from [2, Section 5]. Let us show that axiom (Q2) holds. For $\phi, \psi \in \mathcal{H}_k$, we have that

$$\langle \Pi_k(f\phi), \psi \rangle_k = \langle f\phi, \psi \rangle_k = \int_M h_k(f\phi, \psi) \mu_M$$

because Π_k is self-adjoint and $\Pi_k \psi = \psi$; by sesquilinearity of h_k , this yields

$$\langle \phi, \bar{f}\psi \rangle_k = \int_M h_k(\phi, \bar{f}\psi) \mu_M = \langle \phi, \bar{f}\psi \rangle_k = \langle \phi, \Pi_k(\bar{f}\psi) \rangle_k.$$

This means that $\text{Op}(f)^* = \text{Op}(\bar{f})$. \square

Remark 7.3. We have assumed that M is Kähler for convenience, but there exist ways to construct a Berezin-Toeplitz quantization on a compact symplectic, not necessarily Kähler, manifold (M, ω) with $[\omega/(2\pi)]$ integral, see for instance [3, 18, 8]. \oslash

Assume now that M is endowed with a smooth symplectic, but not Hamiltonian, action of \mathbb{S}^1 . We now identify \mathbb{R}/\mathbb{Z} with the unit circle \mathbb{S}^1 in \mathbb{C} by means of the map $\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{S}^1, [t] \mapsto \exp(2i\pi t)$. Since the symplectic form $\tilde{\omega} = \omega/2\pi$ is integral, there exists a circle valued momentum map $\tilde{\mu}$ with respect to $\tilde{\omega}$ for the action, whose value at $m \in M$ is given by the formula $\tilde{\mu}(m) = \left[\int_{\gamma_m} \mathbf{i}_{1_M} \tilde{\omega} \right]$, where γ_m is a smooth path connecting a fixed point $m_0 \in M$ to m . Hence we get a function $\mu \in \mathcal{C}^\infty(M, \mathbb{S}^1)$ defined as $\mu(m) = \exp(2i\pi \tilde{\mu}(m)) = \exp\left(i \int_{\gamma_m} \mathbf{i}_{1_M} \omega\right)$. We associate to this function a unitary Berezin-Toeplitz operator as follows. Set $V(k) = \text{Op}_k(\mu)$; then $V(k)$ is a Berezin-Toeplitz operator with principal symbol μ but may not be unitary. However, the operator $U(k) := V(k) (V(k)^* V(k))^{-1/2}$ is well-defined, clearly unitary, and it follows from the stability of Berezin-Toeplitz operators with respect to smooth functional calculus [6, Proposition 12] that it is a Berezin-Toeplitz operator with principal symbol μ .

7.3. A family of examples. Following these constructions, we introduce a family of examples for manifolds $M = \mathbb{T}^{2d}$. We start with the case $d = 1$.

An example when $d = 1$. A famous example of symplectic but non Hamiltonian circle action is the action of $S^1 = \mathbb{R}/\mathbb{Z}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by the formula: $[t] \cdot ([q, p]) = ([t + q, p])$. Here the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is endowed with the symplectic form coming from the standard one on \mathbb{R}^2 , that is: $\omega = dp \wedge dq$. The action is clearly symplectic, and is not Hamiltonian, for instance because it has no fixed point.

Lemma 7.4. *The circle-valued momentum map associated with this action is $\tilde{\mu}([q, p]) = [p]$ up to the addition of a constant.*

Proof. Using the notation of the previous section, we have that $\Phi_{[t]}([q, p]) = [t + q, p]$, hence $1_M([q, p]) = \frac{\partial}{\partial q}$, therefore $\mathbf{i}_{1_M}\omega = dp$. Take $m_0 = [0, 0] \in \mathbb{T}^2$ and let $m = [q, p]$ be any point in \mathbb{T}^2 . Then $\gamma_m : [0, 1] \rightarrow \mathbb{T}^2$, $t \mapsto [tq, tp]$ is a smooth path connecting m_0 to m . Thus $\tilde{\mu}(m) = \int_{\gamma_m} dp = \int_0^1 p \, dt = p$. \square

As in the previous part, this map gives rise to a map $\mu \in \mathcal{C}^\infty(\mathbb{T}^2, S^1)$, $\mu([q, p]) = \exp(2i\pi p)$. We have a natural semiclassical operator associated with this momentum map, in the setting of Berezin-Toeplitz quantization. Firstly, let us briefly describe the geometric quantization of the torus, although it is now quite standard (see [20, Chapter I.3] for instance). Let $L_{\mathbb{R}^2} \rightarrow \mathbb{R}^2$ be the trivial line bundle with standard Hermitian form and connection $d - i\alpha$, where α is the 1-form defined as $\alpha_u(v) = \frac{1}{2}\omega(u, v)$, equipped with the unique holomorphic structure compatible with the Hermitian structure and the connection. Consider a lattice $\Lambda \subset \mathbb{R}^2$ of symplectic volume 4π . The Heisenberg group $H = \mathbb{R}^2 \times U(1)$ with product $(x, u) \star (y, v) = (x + y, uv \exp(\frac{i}{2}\omega_0(x, y)))$ acts on $L_{\mathbb{R}^2}$, with action given by the same formula. This action preserves all the relevant structures, and the lattice Λ injects into H ; therefore, by taking the quotient, we obtain a pre-quantum line bundle L over $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$. Furthermore, the action extends to the line bundle $L_{\mathbb{R}^2}^{\otimes k}$ by $(x, u) \cdot (y, v) = (x + y, u^k v \exp(\frac{ik}{2}\omega_0(x, y)))$. We thus get an action $T^* : \Lambda \rightarrow \text{End}(\mathcal{C}^\infty(\mathbb{R}^2, L_{\mathbb{R}^2}^{\otimes k}))$, $u \mapsto T_u^*$. The Hilbert space $\mathcal{H}_k = H^0(\mathbb{T}^2, L^{\otimes k})$ can naturally be identified with the space $\mathcal{H}_{\Lambda, k}$ of holomorphic sections of $L_{\mathbb{R}^2}^{\otimes k} \rightarrow \mathbb{R}^2$ which are invariant under the action of Λ , endowed with the Hermitian product $\langle \phi, \psi \rangle_k = \int_D \phi \bar{\psi} |\omega|$ where D is the fundamental domain of the lattice. Furthermore, $\Lambda/2k$ acts on $\mathcal{H}_{\Lambda, k}$. Let e and f be generators of Λ satisfying $\omega(e, f) = 4\pi$; one can show that there exists an orthonormal basis $(\psi_\ell)_{\ell \in \mathbb{Z}/2k\mathbb{Z}}$ of $\mathcal{H}_{\Lambda, k}$ such that

$$\forall \ell \in \mathbb{Z}/2k\mathbb{Z} \quad \begin{cases} T_{e/2k}^* \psi_\ell = w^\ell \psi_\ell \\ T_{f/2k}^* \psi_\ell = \psi_{\ell+1} \end{cases}$$

with $w = \exp(\frac{i\pi}{k})$. The ψ_ℓ can be computed using Theta functions.

Now, set $U(k) = T_{e/2k}^* : \mathcal{H}_k \rightarrow \mathcal{H}_k$; of course, $U(k)$ is unitary. Let (q, p) be coordinates on \mathbb{R}^2 associated with the basis (e, f) and $[q, p]$ be the equivalence class of (q, p) . It is known [9, Theorem 3.1] that $U(k)$ is a

Berezin-Toeplitz operator with principal symbol $[q, p] \mapsto \exp(2i\pi p)$, which is precisely μ . Trivially, $\text{Sp}(U(k)) = \{\exp(i\pi\ell/k), 0 \leq \ell \leq 2k-1\}$ which is dense in $\mu(\mathbb{T}^2) = \mathbb{S}^1$ when k goes to infinity. Thus, this example is interesting because the assumptions of Theorem 6.6 are not satisfied, since μ is onto, yet we can recover $\mu(M)$ from the spectrum of $U(k)$ when $k \rightarrow +\infty$. This provides with evidence for Conjecture 6.10 in the $d = 1$ case; we will now explain how to do the same in higher dimension.

The higher dimensional case. More generally, we can consider d symplectic but non Hamiltonian circle actions on $M = \mathbb{T}^{2d} = (\mathbb{T}^2)^d$, endowed with the symplectic form coming from $\omega = dp_1 \wedge dq_1 + \dots + dp_d \wedge dq_d$ as follows: for $j \in \llbracket 1, d \rrbracket$, the j -th action is the action of \mathbb{S}^1 described above applied to the j -th copy of \mathbb{T}^2 :

$$[t] \cdot [q_1, p_1, \dots, q_d, p_d] = [q_1, p_1, \dots, q_{j-1}, p_{j-1}, t + q_j, p_j, q_{j+1}, p_{j+1}, \dots, q_d, p_d].$$

This action admits the circle valued moment map $\mu_j \in \mathcal{C}^\infty(\mathbb{T}^{2d}, \mathbb{S}^1)$, where $\mu_j([q_1, p_1, \dots, q_d, p_d]) = \exp(2i\pi p_j)$. Now, we recall the following useful property of Berezin-Toeplitz quantization with respect to direct products: if M_1, M_2 are two compact connected Kähler manifolds endowed with prequantum line bundles L_1 and L_2 respectively, the line bundle

$$L = L_1 \boxtimes L_2 := \pi_1^* L_1 \otimes \pi_2^* L_2 \rightarrow M = M_1 \times M_2$$

is a prequantum line bundle (here $\pi_j : M \rightarrow M_j$ is the natural projection). Moreover, the quantum Hilbert spaces satisfy

$$H^0(M, L^{\otimes k}) = H^0(M_1, L_1^{\otimes k}) \otimes H^0(M_2, L_2^{\otimes k})$$

and, if $f_j \in \mathcal{C}^\infty(M, \mathbb{C})$, $j = 1, 2$, then $\text{Op}_k(f) = \text{Op}_k(f_1) \otimes \text{Op}_k(f_2)$ for $f(m_1, m_2) = f(m_1)f(m_2)$. Coming back to our example where the manifold is $M = \mathbb{T}^2 \times \dots \times \mathbb{T}^2$, we quantize \mathbb{T}^2 as explained in the previous section and we obtain a family of quantum spaces $\mathcal{H}_k = H^0(\mathbb{T}^2, L^{\otimes k})^{\otimes d}$ with orthonormal basis $(\psi_{\ell_1} \otimes \dots \otimes \psi_{\ell_d})_{\ell_1, \dots, \ell_d \in \mathbb{Z}/2k\mathbb{Z}}$. Let $U(k)$ be the same operator as in the previous section, and introduce the operator

$$V_j(k) := \text{Id} \otimes \dots \otimes \text{Id} \otimes \underbrace{U(k)}_{j\text{-th position}} \otimes \dots \otimes \text{Id}$$

for every $j \in \llbracket 1, d \rrbracket$. Then $(V_1(k), \dots, V_d(k))$ is a family of pairwise commuting unitary Berezin-Toeplitz operator acting on \mathcal{H}_k , with joint principal symbol $\mu = (\mu_1, \dots, \mu_d)$. Its joint spectrum is equal to

$$\left\{ \left(\exp\left(\frac{i\pi\ell_1}{k}\right), \dots, \exp\left(\frac{i\pi\ell_d}{k}\right) \right), \ell_1, \dots, \ell_d \in \mathbb{Z}/2k\mathbb{Z} \right\}$$

and again, from this we recover $\mu(M) = \mathbb{T}^d$ when k goes to infinity.

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